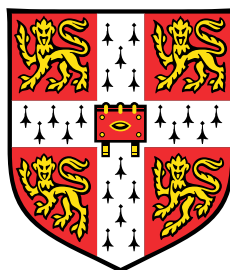


# Études in Ambitwistor Strings

## Exploring new models, higher loops and curved backgrounds



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This dissertation is submitted for the degree of  
*Doctor of Philosophy*

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September 2018



*To my family, with love.*



## Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

Kai Alexander Felix Röhrig  
September 2018



## Acknowledgements

*“If I have seen further than others, it is by standing upon the shoulders of giants.”*

– Isaac Newton

I would like to express my profound gratitude to my supervisor, David Skinner. David has inspired, motivated and guided me starting from my days in part III. He introduced me to the wonders of twistor strings, and the fascination with what he taught me encouraged me to join the field and strive to contribute. His creativity, enthusiasm and depth of understanding make him an extraordinary role model and mentor.

I also owe a great deal to Tim, Piotr, and Eduardo, whose patience and comradeship has made the hard work an enjoyable journey. I learned invaluable lessons from each of them, about physics, academia, and life in general.

I also want to thank Sabrina, for being by my side all these years. She has been a true partner in life, and we have gone through the ups and downs of our PhDs hand-in-hand. This thesis would not have been possible without her, and I look forward to what we will accomplish together next.

Last but not least I want to thank my parents and grandparents for bringing me into this world and putting me on my trajectory with their boundless dedication and love, without ever expecting anything in return. They were present at every step in my life and gave me the tools and the courage to pursue my dreams. Words could never do justice the immense gratitude my sisters and I owe them.





# Abstract

We begin with an overview of the scattering equations, CHY formalism and the ambitwistor string. We discuss one of the striking simplifications that occur upon restricting to four spacetime dimensions, namely that the scattering equations with  $n$  particles decompose into sectors, graded by an integer  $1 \leq d \leq n - 3$ . It is a non-trivial fact that the dimension agnostic CHY formulae reduce to twistor formulae once the external kinematics is restricted to four dimensions. To establish the link between them, we find and prove a formula which describes the splitting of 4-vector-valued fermion correlators on the sphere into a product of two terms, each involving left/right-handed spinors only.

We use this splitting result to derive a formula for  $\mathcal{N} = 4$  super Einstein-Yang-Mills in twistor space based on the refined 4d scattering equations. It computes all tree level amplitudes, in all trace sectors, of minimally coupled sEYM with one gluon multiplet and two, CP conjugate, gravity multiplets. The RSV formula for  $\mathcal{N} = 4$  super-Yang-Mills and a certain subsector of the CS formula for  $\mathcal{N} = 8$  super-gravity are shown to be contained as special cases.

Next, we return to the dimension agnostic setting and present a collection of new ambitwistor string models, which compute the CHY formulae for DBI, Galileons, and several other low energy effective field theories. We describe two attempts at constructing an ambitwistor string for Einstein-Yang-Mills, and why they fail.

In chapter II we initiate a study of the ambitwistor string on a group manifold. After studying the classical theory and quantization on a generic group manifold, we specialize to an  $AdS_3 \times S^3$  background with pure NS-NS flux. We describe how the quantum consistency of the model requires the background and fluctuations to satisfy the supergravity equations of motion, construct explicit vertex operators and discuss correlators. We explore the prospect of a localisation on generalised scattering equations.

In chapter III we present a new operator in the ambitwistor string CFT which allows the computation of amplitudes by gluing together correlators with fewer points or of lower genus. We conjecture it to be the infinite tension limit of the standard

string propagator. Due to the finiteness of the ambitwistor string spectrum, the gluing operator turns out to be a tractable object.

We demonstrate by explicit calculations how our operator underpins the recursive construction of tree-level CHY scattering amplitudes by Dolan & Goddard, as well as the computation of loop integrands on a Riemann sphere by Geyer et al. The gluing operator is schematically a product of two standard ambitwistor vertex operators. It is suitably continued to off-shell momentum while retaining BRST invariance, and intuitively represents the target space Feynman propagator.

We conclude with an exposition of some unsolved problems, open questions, new ideas and aspirations.

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# Introduction

In 2004 Witten [1] brought together two apparently very different attempts to achieve the quantisation of space-time, string theory and twistor theory. He wrote down a string-like formula, which computes the famous Parke-Taylor amplitude [2] for gluon scattering from the moduli integral over holomorphic embeddings of the string world-sheet into super-twistor space. Moreover, it goes beyond the maximally helicity violating case of PT and computes the scattering for any  $\mathcal{N} = 4$  external helicity configuration. The central insight was that, while the MHV amplitude is given by a degree 1 embedding, the  $N^k$ MHV amplitudes arise from degree  $d = k + 1$  embeddings, more precisely, D-instantons. Witten identified the underlying string theory as the topological B-model.

Following along these lines of thought, a formulation similar to that of [1] was found by Cachazo and Skinner [3] for the tree-level scattering of  $\mathcal{N} = 8$  gravitons. In addition to the ingredients of Witten's formula, it contains the Hodges and dual Hodges matrices [4, 5], which in this formalism are nothing but *resultants* of the embedding map [6]. In 2013 Skinner [7] realised this formula as the path integral of a string theory with twistor space as target space. The twistor action supplies a Poisson structure on that target space, which indeed appears in the twistor-string action, and is responsible for breaking conformal symmetry, eventually giving rise to the structure captured by the Hodges matrices. The Penrose transform supplies the fields that govern the deformations of the world-sheet and whose quanta serve as external states. It is worth mentioning that this model has features clearly distinguishing it from classical string theory, for example the absence of the explicit gauge constraint  $T = 0$ .

In parallel to these developments Cachazo, He and Yuan (CHY) drew attention to what they call the scattering equations [8], which provide a map from the space of kinematic invariants of  $n$  massless particles to the moduli space of the sphere with  $n$  punctures. They proposed that these equations might sit at the heart of new formulations of scattering amplitudes in terms of a sum over the  $(n - 3)!$  images of any given kinematic configuration and subsequently delivered these for various field theories

[9, 10]. These formulae enjoy a simple yet rich structure, which can be captured in a universal summation/integration measure, supplemented by a theory dependent summand/integrand.

One striking feature, known as *KLT orthogonality* [8], of the map provided by the scattering equations, is that the images of a kinematic point can be further mapped to vectors in the space of certain chiral conformal blocks, which are orthogonal with respect to the KLT bilinear form [11]. This allows the CHY formulae to be written as inner products, upon which the KLT relations [12] between gauge and gravity amplitudes become manifest and the KLT kernel becomes associated with the interactions of a coloured cubic scalar.

The most notable difference compared to the Witten and CS twistor-string formulae is that the CHY formulae are valid in any dimension. In particular, they make no reference to spinors or supersymmetry. Nevertheless, the formulation in terms of a punctured sphere is suggestive of an underlying string theory. This theory was discovered by Mason and Skinner [13] for the gravity CHY formula and generalises it to scattering the full gravity supermultiplet. Its target space is the super-ambitwistor space; it is formulated in a priori arbitrary dimension and the world-sheet central charge vanishes only in ten dimensions, ensuring modular invariance and consistency of the  $\mathcal{N} = 2$  SUSY (à la RNS). The images of the scattering map arise as localisation locus of the world-sheet path integral [14], providing the universal summation measure, while world-sheet fermions provide the theory-specific summand.

A significant advantage of having at hand a worldsheet theory which gives rise to the CHY formulae is that it provides a recipe for computing higher loop amplitudes and scattering on non-trivial backgrounds. Both of these exciting directions are being actively pursued by the community [15–21], and this thesis is a record of the author’s efforts devoted to this cause.

Another notable feature of the scattering equations and the CHY formulae is the way in which they manifest factorisation and universal soft theorems [22, 8, 23]. The scattering equations provide a link between the factorisation of target space interactions and the worldsheet geometry, which may be attributed to the string theory heritage. On the other hand, they provide a strikingly clear tool for understanding soft theorems, which appears to be rather specific to the ambitwistor formulation. In this spirit, studies of the soft theorems and their relation to the BMS group [24, 25] have been performed using related world-sheet models [26–30]. These properties are the silent heroes of many results and calculations, as they impose constraints which can be used to construct, inspire and check ansätze.

Some of the work presented in this thesis was done in collaboration with other researchers.

The work in section 1.5 was done in collaboration with Eduardo Casali, Yvonne Geyer, Lionel Mason and Riccardo Monteiro and was published in [31]. The work in chapter 3 was done in collaboration with David Skinner and was published in [32]. The work in chapter 2 is an ongoing collaboration with David Skinner and is not yet published.

The work in sections 1.3 and 1.4 is the original work of the author and was published in [33].

Most of sections 1.1 and 1.2 contains reviews of known results from the literature, which are cited accordingly.





# Chapter 1

## Ambitwistor Strings at Tree Level

### 1.1 The Scattering Equations

At the heart of both the CHY framework and the connected prescription twistor string formulas lie the scattering equations: a map between the kinematic configuration space of massless particles, and the moduli space of Riemann surfaces. They were first discovered by Fairlie and Roberts [34–36] in the context of high energy scattering in string theory, where they arise in an application of Laplace’s method to the moduli space integral as the locus of the dominant contribution to the amplitude. From this perspective, their possibly most natural form is

$$dS = 0 \ , \quad S = \sum_{\substack{i,j=1 \\ i \neq j}}^n p_i \cdot p_j \log(z_i - z_j) \quad (1.1)$$

where  $p_i$  are the on-shell momenta of  $n$  external massless particles and  $z_i$  the corresponding insertion points on the string worldsheet, while  $d$  denotes the exterior derivative on the moduli space of the  $n$ -punctured Riemann sphere. We remark that  $S$  is the holomorphic part of the exponent in the Koba-Nielson factor.

One important insight of [37] was that the scattering equations can be reformulated as the requirement that the holomorphic differential

$$P_\mu(z) := \sum_{i=1}^n (p_i)_\mu \frac{dz (z_i - z_*)}{(z - z_i)(z - z_*)} \ , \quad (1.2)$$

where  $z_*$  is an arbitrary auxiliary point on the sphere, and  $P(z)$  is independent of  $z_*$  by momentum conservation, should be null

$$P^2(z) = 0 \tag{1.3}$$

everywhere on the sphere.<sup>1</sup> Since  $P$  is a meromorphic  $(1,0)$ -form,  $P^2$  contains  $n - 3$  independent components: we can use Cauchy's theorem to write the value of the meromorphic quadratic differential  $P^2(z)$  at any  $z$  in terms of its values at some fixed  $n - 3$  points.

Indeed, due to Moebius invariance, the moduli space of the Riemann sphere is  $n - 3$  dimensional, so eqs. (1.1) and (1.3) contain  $n - 3$  independent equations. In standard local coordinates for the moduli space they read

$$\sum_{\substack{j=1 \\ j \neq i}}^n p_i \cdot p_j \frac{(z_j - z_*)}{(z_i - z_j)(z_i - z_*)} = 0, \quad \text{for } i = 1, \dots, n - 3. \tag{1.5}$$

The most fundamental characteristic of these equations is that they are rational and generically have  $(n - 3)!$  solutions [8, 38].

While the representations (1.1),(1.3) are appealing and suggestive, there are many more representations of the same equations, each equipped with their own interpretation. It is one of the goals of the present work, and one of the principal personal motivations for the author, to explore the relationship between these as far as possible, in pursuit of an understanding of the fundamental role they play in quantum field theory. The properties of these equations and their solutions have been studied widely and deeply (c.f. [38–44] and many more) and the present work draws on many of these results, so we will review many of them and even add some new ones.

### 1.1.1 Representations of the refined Scattering Equations

It is well known that in four dimensions the scattering equations split into R-charge sectors, also known as  $N^{d-1}$ MHV sectors. These sectors are labelled by an integer  $d$ , or  $\tilde{d} \equiv n - d - 2$ , where  $n$  is the total number of particles. There are many equivalent representations of these refined scattering equations, and we will now briefly recall those

---

<sup>1</sup>The connection to the previous perspective is that

$$dS = \sum_i \text{Res}_{z_i} P^2 \tag{1.4}$$

as one-forms on the moduli space.

three which we use throughout the present work. The idea is to write the particles' momenta as matrices in spinor-helicity notation by contracting them with the Pauli matrices

$$p_\mu (\sigma^\mu)^{\alpha\dot{\alpha}} \equiv p^{\alpha\dot{\alpha}} \quad (1.6)$$

with the essential property that  $p^2 = \det p$ , so that a null momentum can be factorised as  $p = \lambda \otimes \tilde{\lambda}$ . Then we can solve the scattering equations

$$\det P = 0 \ , \quad (1.7)$$

where

$$P^{\alpha\dot{\alpha}}(z) := \sum_{i \in \mathbf{p}} \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \frac{dz (z_i - z_*)}{(z - z_i)(z - z_*)} \ , \quad (1.8)$$

by factorizing [37] it as

$$P^{\alpha\dot{\alpha}}(z) = \lambda^\alpha(z) \tilde{\lambda}^{\dot{\alpha}}(z) \ , \quad (1.9)$$

globally on the sphere. Here we introduced the shorthand  $\mathbf{p} = \{1, \dots, n\}$  for the set of all particle labels, which will be used throughout this work. The factorisation involves a choice of how to distribute the zeros of  $P(z)$  as a function of  $z$  among the two factors, and this choice labels the different refinement sectors. It also requires a choice of how to distribute the poles of  $P(z)$  among the two factors and this choice labels the various equivalent representations of the scattering equations.

The first representation of the refined scattering equation is given by the splitting

$$P(z) = \lambda_T(z) \tilde{\lambda}_T(z) \quad (1.10)$$

with

$$\lambda_T^\alpha \in H^0(\mathcal{O}(d)) \ , \quad \tilde{\lambda}_T^{\dot{\alpha}} \in H^0\left(\mathcal{O}(-d) \otimes K\left[\sum_{i \in \mathbf{p}} z_i\right]\right) \ , \quad (1.11)$$

where the subscript stands for *twistor*. The notation here means that for  $\alpha = 0, 1$ ,  $\lambda_T^\alpha$  is a holomorphic polynomial of degree  $d$  while  $\tilde{\lambda}_T^{\dot{\alpha}}$  is a meromorphic  $(1, 0)$ -form of homogeneity  $-d$  with simple poles at all marked points. In these variables the scattering equations read

$$\text{Res}_{z_i} \tilde{\lambda}_T = t_i \tilde{\lambda}_i \ , \quad t_i \lambda_T(z_i) = \lambda_i \quad \forall i \in \mathbf{p} = \{1, \dots, n\} \ . \quad (1.12)$$

They fix the sections  $\lambda_T, \tilde{\lambda}_T$ , the scaling parameters<sup>2</sup>  $t_i$  and locations  $z_i$  (up to Möbius invariance), and also enforce momentum conservation.

The distinct refinement sectors are labelled by the integer  $d$ , and the original scattering equations  $P^2 = 0$  are equivalent to the union of the refined scattering equations for  $d = 1, \dots, n - 3$ . Each sector contains  $A(n - 3, d - 1)$  solutions, where  $A(p, q)$  denote the Eulerian numbers<sup>3</sup>; the summation identity  $\sum_{q=0}^{p-1} A(p, q) = p!$  guarantees that the total number of solutions is preserved, and the reflection property  $A(p, q) = A(p, p - q - 1)$  ensures that parity can act as an involution on the solutions.

The second representation is the parity conjugate of the previous one and is given by the splitting

$$P(z) = \lambda_{\tilde{T}}(z) \tilde{\lambda}_{\tilde{T}}(z) \quad (1.13)$$

with

$$\lambda_{\tilde{T}}^\alpha \in H^0 \left( \mathcal{O}(-\tilde{d}) \otimes K \left[ \sum_{i \in \mathbf{p}} z_i \right] \right), \quad \tilde{\lambda}_{\tilde{T}}^\alpha \in H^0 \left( \mathcal{O}(\tilde{d}) \right), \quad (1.14)$$

where the subscript stands for *dual twistor*. Here  $\lambda_{\tilde{T}}$  is a meromorphic  $(1, 0)$ -form of homogeneity  $-\tilde{d}$  with simple poles at all marked points, while  $\tilde{\lambda}_{\tilde{T}}$  is a holomorphic polynomial of degree  $\tilde{d}$ . In these variables the scattering equations read

$$\text{Res}_{z_i} \lambda_{\tilde{T}} = \tilde{t}_i \lambda_i, \quad \tilde{t}_i \tilde{\lambda}_{\tilde{T}}(z_i) = \tilde{\lambda}_i \quad \forall i \in \mathbf{p} \quad (1.15)$$

and they again fix the sections  $\lambda_{\tilde{T}}, \tilde{\lambda}_{\tilde{T}}$ , the scaling parameters  $\tilde{t}_i$  and locations  $z_i$  (up to Möbius invariance) and enforce momentum conservation.

The third representation is useful if there is a natural splitting of the set of external particles  $\mathbf{p}$  into two subsets  $\mathbf{p}^+ \cup \mathbf{p}^- = \mathbf{p}$ , such as positive/negative helicity. Then we can require that

$$P(z) = \lambda_A(z) \tilde{\lambda}_A(z) \quad (1.16)$$

with

$$\lambda_A^\alpha \in H^0 \left( K^{1/2} \left[ \sum_{i \in \mathbf{p}^-} z_i \right] \right), \quad \tilde{\lambda}_A^\alpha \in H^0 \left( K^{1/2} \left[ \sum_{i \in \mathbf{p}^+} z_i \right] \right), \quad (1.17)$$

---

<sup>2</sup>Both  $\lambda_T(z)$  and  $\lambda_i$  are only defined up to rescaling by a non-zero complex number. Hence the scattering equations can only require them to be proportional, and the scaling parameters  $t_i$  are introduced to account for the rescaling covariance.

<sup>3</sup>The Eulerian number  $A(p, q)$  is the number of permutations of 1 to  $p$  where  $q$  elements are larger than their preceding element. They are defined recursively by  $A(p, q) = (p - q)A(p - 1, q - 1) + (q + 1)A(p - 1, q)$ .

where the subscript stands for *ambi-twistor*. Here  $\lambda_A, \tilde{\lambda}_A$  are both meromorphic  $(1/2, 0)$ -forms of homogeneity 0 and have simple poles at the marked points in  $p^-, p^+$  respectively. In these new variables the scattering equations read

$$\begin{aligned} \text{Res}_{z_i} \lambda_A &= \tilde{u}_i \lambda_i, & \tilde{u}_i \tilde{\lambda}_A(z_i) &= \tilde{\lambda}_i & \forall i \in p^- \\ \text{Res}_{z_i} \tilde{\lambda}_A &= u_i \tilde{\lambda}_i, & u_i \lambda_A(z_i) &= \lambda_i & \forall i \in p^+ \end{aligned} \quad (1.18)$$

and they fix the sections  $\lambda_A, \tilde{\lambda}_A$ , the scaling parameters  $u_i, \tilde{u}_i$  and locations  $z_i$  (up to Möbius invariance) and enforce momentum conservation. We note that this representation of the scattering equations enables a manifestly parity symmetric reformulation of the RSVW and CS formulae [45, 46].

We can easily switch between these three representation via the relations

$$\begin{aligned} \lambda_T(z) &\propto \frac{\prod_{i \in p^-} (z - z_i)}{\sqrt{dz}} \lambda_A(z) \propto \frac{\prod_{i \in p} (z - z_i)}{dz} \lambda_{\tilde{T}}(z) \\ \tilde{\lambda}_T(z) &\propto \frac{\sqrt{dz}}{\prod_{i \in p^-} (z - z_i)} \tilde{\lambda}_A(z) \propto \frac{dz}{\prod_{i \in p} (z - z_i)} \tilde{\lambda}_{\tilde{T}}(z) \end{aligned} \quad (1.19)$$

for the sections, where the factor of proportionality is independent of  $z$ , and also for the scaling parameters

$$\frac{t_j}{t_i} \prod_{k \in p^- \setminus \{i\}} \frac{z_j - z_k}{z_i - z_k} = \tilde{u}_i u_j \frac{\sqrt{dz_i dz_j}}{z_i - z_j} = \frac{\tilde{t}_i}{\tilde{t}_j} \prod_{k \in p^+ \setminus \{j\}} \frac{z_i - z_k}{z_j - z_k} \quad (1.20)$$

for any choice of  $i \in p^-, j \in p^+$ . The locations  $z_i$  are identical among the three representations.

We highlight that among the three representations the number of zeros in  $\lambda_A, \lambda_T, \lambda_{\tilde{T}}$  and  $\tilde{\lambda}_A, \tilde{\lambda}_T, \tilde{\lambda}_{\tilde{T}}$  is always  $d$  and  $\tilde{d}$  respectively, and only the poles are redistributed. Of course one may define many more representations of the same equations by choosing different ways of distributing the poles among the two factors.

## 1.2 A brief review of the ambitwistor string

The formulae of Cachazo, He and Yuan give rise to expressions for scattering amplitudes as a sum over solutions to the scattering equations of certain rational functions. They give tree amplitudes in the general form of an integral over the moduli space of the

$n$ -punctured Riemann sphere

$$\mathcal{A}_n = \delta^d \left( \sum_{i=1}^n p_i \right) \int_{\mathcal{M}_{0,n}} \bar{\delta}^{(n-3)}(P^2) \mathcal{I}^{(L)} \mathcal{I}^{(R)}$$

where  $z_i$ ,  $i = 1, \dots, n$  are complex coordinates of each puncture of the Riemann spheres,  $p_i$  the null momenta of the massless particles in the scattering process, and  $P^2 = 0$  are the scattering equations formulated in terms of the holomorphic differential  $P(z)$  (1.2). They are imposed by the delta functions

$$\bar{\delta}(z) = \bar{\partial} \frac{1}{2\pi i z} = \delta(\Re z) \delta(\Im z) d\bar{z},$$

where  $\Re$  denotes the real part and  $\Im$  the imaginary part, impose the *scattering equations*:  $k_i \cdot P(\sigma_i) = 0$ . Thus, the integral over the moduli space reduces to a sum over solutions to the scattering equations of the integrand, multiplied by a Jacobian factor. The integrand naturally decomposes into two factors  $\mathcal{I}^{(L)}$  and  $\mathcal{I}^{(R)}$ , sometimes called ‘half-integrands’, that depend on the locations  $z_i$ , as well as the quantum numbers such as momentum, polarisation and/or colour data of the particles whose scattering is being computed, and depends on the theory. The  $\mathcal{I}^{(L)}$  and  $\mathcal{I}^{(R)}$  are meromorphic one-forms on the moduli space, and can be chosen from five different choices and the various theories arise from the different possible combinations.

While the integrand of the CYH formula (1.2) is familiar from standard string theory amplitude, it differs in that *both* ‘half-integrands’ are left-moving.

In [13] Mason and Skinner constructed a world-sheet CFT model, called ambitwistor string theory, which computes the CHY formula for gravity. Ambitwistor strings are chiral infinite tension analogues of RNS strings that can be interpreted, after reduction of constraints, as strings whose target space is the space of complexified null geodesics in Minkowski space. This space of complexified null geodesics has become known as ambitwistor space.

We begin by briefly reviewing the salient points in the construction of the ambitwistor string. Further details may be found in [13, 26, 14]. All ambitwistor strings are based on the chiral, bosonic action

$$S = \int_{\Sigma} P \bar{\partial} X - \frac{e}{2} P^2 \quad (1.21)$$

where  $X : \Sigma \rightarrow M$  and  $P \in \Omega^{1,0}(\Sigma, T^*M)$ . For the purposes of this chapter we take  $M = \mathbb{C}^D$  to be (the complexification of)  $D$ -dimensional flat space, though the ambitwistor string can also be placed on a curved background, see [47, 16, 17] and

chapter 2. The field  $e \in \Omega^{0,1}(\Sigma, T_\Sigma)$  transforms like a Beltrami differential on  $\Sigma$  and acts as a Lagrange multiplier enforcing the constraint

$$P^2(z) = 0, \quad \forall z \in \Sigma, \quad (1.22)$$

so that  $P_\mu$  must be null. This constraint generates the gauge transformations

$$X^\mu \mapsto X^\mu + \alpha \eta^{\mu\nu} P_\nu \quad e \mapsto e + \bar{\partial}\alpha \quad (1.23)$$

where  $\alpha$  is a smooth  $(1,0)$ -vector on  $\Sigma$ , whilst  $P_\mu$  itself remains invariant. Thus  $X$  is defined modulo translation along lightlike directions and, accounting for this redundancy, the moduli space of the ambitwistor string is the space of light rays in  $M$ , known as *ambitwistor space*.

Aside from the action, the only other occurrence of the field  $X(z)$  comes from the vertex operators. For plane waves of momentum  $p_i$ , the  $X$  path integral is thus

$$\int \mathcal{D}X \, e^{-\int P \bar{\partial} X} \prod_{i=1}^n e^{i p_i \cdot X(z_i)} \quad (1.24)$$

in the presence of  $n$  such vertex operators. Integrating out the zero mode of  $X$  leads to a momentum-conserving  $\delta$ -function, whilst integrating out the non-zero modes of  $X$  freezes the quantum field  $P(z)$  to its classical value

$$P(z) = \sum_{i=1}^n p_i \, \omega_{i*}(z), \quad (1.25)$$

where

$$\omega_{i*}(z) \equiv \frac{dz (z_i - z_*)}{(z - z_i)(z - z_*)} \quad (1.26)$$

is the unique meromorphic one-form on the Riemann sphere with simple poles at  $z_i$  and  $z_*$ , with residues  $\pm 1$ , respectively. By overall momentum conservation,  $P(z)$  in (1.25) is in fact independent of the auxiliary reference point  $z_*$ . (The introduction of  $z_*$  is required to make the kinetic operator of the  $P, X$  system invertible and thus define a propagator.)

As explained in [13], the vertex operators lie in the BRST cohomology only if  $p_i^2 = 0$ , so the constraint (1.22) becomes

$$P^2(z) = \sum_{i \neq j} p_i \cdot p_j \, \omega_{i*}(z) \, \omega_{j*}(z) = 0 \quad (1.27)$$

which constrains the location of the punctures in terms of the external momenta, i.e., it says that the worldsheet CFT correlator is only supported at certain special points in  $\overline{\mathcal{M}}_{0,n}$ . Since any meromorphic quadratic differential on a Riemann sphere must have at least 4 poles (counted with multiplicity), the requirement that  $P^2(z) = 0$  throughout  $\Sigma$  can be enforced by asking

$$\text{Res}_{z=z_i} P^2(z) = 2 \sum_{j \neq i} p_i \cdot p_j \omega_{j*}(z_i) = 0 \quad (1.28)$$

at (any)  $n - 3$  of the  $n$  punctures. These are the *scattering equations* [34, 48, 49, 9]. They arise in the ambitwistor string from a careful treatment of the gauge fixing of (1.23); see [26, 14] for further details. (They can also be seen as arising from the ambitwistor cohomology classes used to represent the external states [13].)

### 1.2.1 CHY formulae from the ambitwistor string

In fact, by itself, the bosonic theory above suffers from various anomalies, as may be expected since the worldsheet theory is chiral. One possibility to cure these is to add two sets of fermions, *both* of which are left-moving:

$$\psi \in \Omega^0(\Sigma, K_\Sigma^{1/2} \otimes TM) \quad \text{and} \quad \tilde{\psi} \in \Omega^0(\Sigma, K_\Sigma^{1/2} \otimes TM) .$$

These fields have exactly the same worldsheet quantum numbers, but are subject to independent GSO projections. These fields have action

$$S[\psi, \tilde{\psi}] = \int_\Sigma \frac{1}{2} \psi \cdot \bar{\partial} \psi + \frac{1}{2} \tilde{\psi} \cdot \bar{\partial} \tilde{\psi} - \chi P \cdot \psi - \tilde{\chi} P \cdot \tilde{\psi} \quad (1.29)$$

where the fields  $\chi, \tilde{\chi} \in \Omega^{0,1}(\Sigma, T_\Sigma^{1/2})$  impose constraints  $P \cdot \psi = P \cdot \tilde{\psi} = 0$ . This ambitwistor string is thus very similar to the RNS superstring, but note that *i*) the theory is chiral, with both sets of fermions living in  $K_\Sigma^{1/2}$  and *ii*) whilst the algebra of constraints  $\{P^2, P \cdot \psi, P \cdot \tilde{\psi}\}$  is similar to that of worldsheet supersymmetry in the RNS string, here  $P$  is independent of  $X$  (in particular,  $P \neq \partial X$ ) so the transformations they generate have nothing to do with worldsheet diffeomorphisms; rather, the model is a form of worldsheet gauge theory with gauge supergroup  $PSL(1|1; \mathbb{C})$ . (A pure spinor version of the ambitwistor string has been constructed in [50].)

We BRST quantise this theory by introducing ghost/anti-ghost pairs, along with Nakanishi-Lautrup fields  $H, G, \tilde{G}$ , for each of the gauge symmetries and adding the



gauge fixing term

$$S_{\text{FP}} = \int_{\Sigma} \left\{ Q, \tilde{b}(e - e_0) + \beta(\chi - \chi_0) + \tilde{\beta}(\tilde{\chi} - \tilde{\chi}_0) \right\} \quad (1.30)$$

to the action, where  $e_0, \chi_0, \tilde{\chi}_0$  are the coordinates on moduli space, i.e. the directions of field space transverse to the gauge orbits. The moduli space is finite dimensional and is the remainder of the original path integral after gauge fixing. The BRST operator  $Q$  is defined to act as

$$\begin{aligned} Q \circ (X, P) &= (\tilde{c}P + \gamma\psi + \tilde{\gamma}\tilde{\psi}, 0), \\ Q \circ (\psi, \tilde{\psi}) &= (-\gamma P, -\tilde{\gamma}P), \\ Q \circ e &= \bar{\partial}\tilde{c} - 2\gamma\chi - 2\tilde{\gamma}\tilde{\chi}, \\ Q \circ (\chi, \tilde{\chi}) &= (\bar{\partial}\gamma, \bar{\partial}\tilde{\gamma}), \\ Q \circ \tilde{c} &= \gamma^2 + \tilde{\gamma}^2, \quad Q \circ (\gamma, \tilde{\gamma}) = 0, \\ Q \circ \tilde{b} &= H, \quad Q \circ (\beta, \tilde{\beta}) = (G, \tilde{G}), \\ Q \circ (H, G, \tilde{G}) &= 0. \end{aligned} \quad (1.31)$$

To obtain the correct measure on moduli space we let  $Q$  act as the exterior derivative on moduli space [51]

$$Q \circ e_0 = de_0, \quad Q \circ (\chi_0, \tilde{\chi}_0) = (d\chi_0, d\tilde{\chi}_0). \quad (1.32)$$

Note that this assigns ghost number one and odd Grassmann statistics to the exterior derivative. For the purpose of doing computations using the unfixed action it is convenient to split it into a free-, interaction-, and moduli-part as

$$\begin{aligned} S_1 &= \int_{\Sigma} P \bar{\partial}X + \frac{1}{2} \psi \bar{\partial}\psi + \frac{1}{2} \tilde{\psi} \bar{\partial}\tilde{\psi} - \tilde{b} \bar{\partial}\tilde{c} + \beta \bar{\partial}\gamma + \tilde{\beta} \bar{\partial}\tilde{\gamma} + H e + G \chi + \tilde{G} \tilde{\chi} \\ S_2 &= \int_{\Sigma} -\frac{e}{2} P^2 - \chi \psi \cdot P - \tilde{\chi} \tilde{\psi} \cdot P + 2\tilde{b} \gamma \chi + 2\tilde{b} \tilde{\gamma} \tilde{\chi} \\ S_3 &= \int_{\Sigma} -H e_0 + \tilde{b} de_0 - G \chi_0 - \beta d\chi_0 - \tilde{G} \tilde{\chi}_0 - \tilde{\beta} d\tilde{\chi}_0 \end{aligned} \quad (1.33)$$

and treat the interactions perturbatively when necessary. Following standard BRST quantisation, the BRST charge  $Q$  is given by

$$Q = \oint cT + \tilde{c}H + \gamma G + \tilde{\gamma} \tilde{G} - (\gamma^2 + \tilde{\gamma}^2) \tilde{b}, \quad (1.34)$$

and the action on the fields eq. (1.31) follows from the propagators given by  $S_1$  as well as the interactions provided by  $S_2$ . This BRST operator obeys  $Q^2 = 0$  quantum mechanically iff the target space  $M$  has dimension  $d = 10$ .

Altogether, after integrating out the gauge fields, the *effective* BRST operator for this ambitwistor string is

$$Q_{\text{eff}} = \oint c T + \frac{\tilde{c}}{2} P^2 + \gamma \psi \cdot P + \tilde{\gamma} P \cdot \tilde{\psi} - (\gamma^2 + \tilde{\gamma}^2) \tilde{b} \quad (1.35)$$

where  $T = P \cdot \partial X + \dots$  is the worldsheet stress tensor,  $c$  is the usual ghost for worldsheet diffeomorphisms, and  $\tilde{c}, \gamma, \tilde{\gamma}$  are ghosts associated to the symmetries generated by the constraints above. Later in this study we will find that the presence of the gluing operator alters the form of this effective BRST operator; we would like to emphasise however that the original form of the operator (1.34) remains valid throughout.

In [13, 26] it was shown that the BRST cohomology consists of only massless states. Fixed vertex operators in the BRST cohomology and surviving the GSO projections take the form

$$\mathcal{O}(z) = c(z) \tilde{c}(z) V(z) \tilde{V}(z) e^{i p \cdot X(z)}, \quad (1.36a)$$

where

$$V_{\text{NS}}(z) = \delta(\gamma(z)) \varepsilon \cdot \psi(z) \quad (1.36b)$$

in the Neveu-Schwarz sector and

$$V_{\text{R}}(z) = e^{-\phi(z)/2} \zeta^\alpha \Theta_\alpha(z) \quad \text{or} \quad V_{\text{R}}(z) = e^{-\phi(z)/2} \bar{\zeta}_{\dot{\alpha}} \Theta^{\dot{\alpha}}(z) \quad (1.36c)$$

in the Ramond sector. Here,  $\varepsilon_\mu$  is a polarisation vector while  $\zeta, \bar{\zeta}$  are left/right handed polarisation spinors, and  $\phi(z)$  is part of the bosonisation of the  $\beta\gamma$  ghost system (see e.g. [52] for details). Note that the spin fields  $\Theta(z)$  carry holomorphic conformal weight  $5/8$  while  $e^{q\phi}$  carries holomorphic conformal weight  $-q(q/2 + 1)$ . After imposing a GSO projection, the spectrum is that of  $D = 10$  Type II A/B supergravity [26], according to whether the untilded and tilded Ramond sectors are chosen to have opposite/same space-time chiralities.

The CHY formula [9] for  $n$ -particle tree-level scattering amplitudes in gravity follows from the genus zero correlator of  $n$  vertex operators in the NS sector, together with the appropriate integrals over the (bosonic and fermionic) worldsheet moduli space. In particular, each of the two CHY Pfaffians arises from the correlator of the fermions in the  $n$  vertex operators, together with the fermions in the  $n - 2$  picture changing

operators that can be viewed as coming from integrating over the fermionic moduli space. Thus, for the  $\psi$ s, we have

$$\left\langle \prod_{i=1}^n \varepsilon \cdot \psi(z_i) \prod_{r=1}^{n-2} \psi(x_r) \cdot P(x_r) \right\rangle = \text{Pf} \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \quad (1.37a)$$

with the sub-matrices given by

$$\begin{aligned} A_{rs} &= P(x_r) \cdot P(x_s) S(x_r, x_s) & \text{for } r \neq s, & & A_{rr} &= 0, \\ B_{ij} &= \varepsilon_i \cdot \varepsilon_j S(z_i, z_j) & \text{for } i \neq j, & & B_{ii} &= 0, \\ C_{ir} &= \varepsilon_i \cdot P(x_r) S(z_i, x_r), \end{aligned} \quad (1.37b)$$

where  $S(z_i, z_j)$  is the genus zero Szegő kernel

$$S(z_i, z_j) \equiv \frac{\sqrt{dz_i dz_j}}{z_i - z_j}.$$

We recall that the  $X$  path integral has frozen the field  $P(z)$  to its classical value (1.25). This fermion correlator is accompanied by the correlator of the ghost insertions

$$\left\langle \prod_{i=1}^n \delta(\gamma(z_i)) \prod_{r=1}^{n-2} \delta(\beta(x_r)) \right\rangle = \frac{\prod_{i<j=1}^n S(z_i, z_j) \prod_{r<s=1}^{n-2} S(x_r, x_s)}{\prod_{i=1}^n \prod_{r=1}^{n-2} S(z_i, x_r)} \quad (1.37c)$$

so the ‘half-integrand’ is

$$\mathcal{I}^{(L/R)} = \frac{\prod_{i<j=1}^n S(z_i, z_j) \prod_{r<s=1}^{n-2} S(x_r, x_s)}{\prod_{i=1}^n \prod_{r=1}^{n-2} S(z_i, x_r)} \text{Pf} \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \quad (1.38)$$

On the support of the scattering equations, Liouville’s theorem implies that (1.38) is independent of the locations  $x_r$ , as expected for PCOs. In particular, we can take the auxiliary points  $x_r$  to coincide with  $n - 2$  of the external punctures, upon which we recover the original Pfaffian of CHY [9]. One can take this limit already at the level of the CFT correlator, changing  $n - 2$  of the picture number  $-1$  vertex operators to

$$\lim_{x \rightarrow z} (\delta(\beta) P \cdot \psi)(x) \left( \delta(\gamma) \varepsilon \cdot \psi e^{ip \cdot X} \right)(z) = \left( (\varepsilon \cdot P + \varepsilon \cdot \psi p \cdot \psi) e^{ip \cdot X} \right)(z) \quad (1.39)$$

of picture number 0. A similar correlation function for the  $\tilde{\psi}$  system produces the second Pfaffian in the CHY formula for gravity amplitudes.

At present, there is no known, consistent ambitwistor string describing Einstein–Yang–Mills theory (nor pure Yang–Mills theory). However, one can generate CHY formulae for tree amplitudes in these theories from anomalous ambitwistor strings, provided one is willing to discard certain terms ‘by hand’. For example, if we replace the  $\tilde{\psi}$  fermion system by a general worldsheet current algebra  $j^a(z)$  obeying

$$j^a(z) j^b(0) \sim \frac{k}{z^2} \delta^{ab} + \frac{1}{z} f_c^{ab} j^c(0) + \mathcal{O}(z^0) , \quad (1.40)$$

then the NS vertex operator

$$\mathcal{O}_{\text{NS}}(z) = c(z) \tilde{c}(z) V_{\text{NS}}(z) t_a j^a(z) e^{ip \cdot X(z)} \quad (1.41a)$$

describes a gluon of polarisation  $\epsilon_\mu$  and colour  $t_a$ , while R sector vertex operator

$$\mathcal{O}_{\text{R}}(z) = c(z) \tilde{c}(z) V_{\text{R}}(z) t_a j^a(z) e^{ip \cdot X(z)} \quad (1.41b)$$

describes a gluino of spin  $\zeta$  and colour  $t_a$ . As shown in [13], inserting  $n$  gluon vertex operators and keeping only the leading trace contribution yields the CHY formula for tree-level scattering in Yang–Mills theory [9], with one Pfaffian arising from the  $\psi$ s as before, and the leading–trace current correlator giving a worldsheet Parke–Taylor factor

$$\text{PT}(\alpha) = \text{tr} \left( t_{\alpha(1)} \cdots t_{\alpha(n)} \right) \prod_{i=1}^n S(z_{\alpha(i)}, z_{\alpha(i+1)}) , \quad (1.42)$$

summed over all inequivalent colour orderings  $\alpha$ , and indices taken modulo  $n$ . Similarly, replacing both  $\psi$  and  $\tilde{\psi}$  by two, independent current algebras  $j_a(z)$  and  $\tilde{j}_{\tilde{a}}(z)$  and inserting vertex operators

$$\mathcal{O}(z) = c(z) \tilde{c}(z) t_a j^a(z) \tilde{t}_{\tilde{a}} \tilde{j}^{\tilde{a}}(z) e^{ip \cdot X(z)} , \quad (1.43)$$

the (double) leading trace terms yield the CHY formula for the cubic, bi-adjoint scalar theory, with two Parke–Taylor factors summed over independent colour orderings  $\alpha$  and  $\beta$ :

$$\mathcal{M}_{\text{bi-adjoint}} = \delta^D \left( \sum_{i=1}^n p_i \right) \sum_{\alpha, \beta \in S_n / \mathbb{Z}_n} \mathbf{m}(\alpha, \beta) , \quad (1.44)$$

where the colour-ordered partial amplitudes

$$\mathbf{m}(\alpha, \beta) = \int \left( \frac{1}{\omega_{123}} \right)^2 \prod_{i=4}^n \bar{\delta} \left( \text{Res}_{z_i} P^2 \right) \text{PT}(\alpha) \text{PT}(\beta) , \quad (1.45)$$

and  $\omega_{123} = S(z_1, z_2) S(z_2, z_3) S(z_3, z_1)$  is the Möbius volume factor.

## 1.3 Refinement in 4d

It is a rather non-trivial fact that the CHY formulae reduce to the corresponding twistor formulae once the external kinematics is four-dimensional. Both are underpinned by the same set of equations, albeit in very different representations, which is widely understood [53, 37, 6, 38, 46, 54] and we review briefly in section 1.1.1. However, the functions on the moduli space which determine the states and interactions look rather different in the CHY and twistor representations.

One of the most striking simplifications that occur in four-dimensional formulation is that the  $(n-3)!$  solutions of the scattering equations group into sectors, labelled by the degree  $1 \leq d \leq n-3$ , as described in section 1.1.1 [38, 39]. What's more, the scattering amplitudes of several theories only receive a contribution from one particular sector. For example, it is well known that  $4d$  scattering amplitudes of Yang-Mills can be organised by MHV sector [2, 55], which counts the number of states of one helicity, and is independent of the number of states of the other helicity. The remarkable simplicity of the maximally helicity violating amplitudes can be traced back to the integrability properties of the underlying (anti-self-dual) field equations, and the higher  $N^k$ MHV amplitudes are a perturbative expansion around this integrable sector. While this perspective breaks manifest parity invariance, it retains a natural action of parity, and the emergence of parity invariance is understood [37, 53]. After incorporation of supersymmetry, the MHV sectors are generalised to  $R$ -charge super-selection sectors.

It was one of the central insights by Witten that the scattering amplitudes of super-Yang-Mills in the  $R$  charge sector  $k$  are supported on curves in twistor space of degree  $d = k + 1$ , and, equivalently, receive contributions only from solutions in the sector  $d$ . While this statement is entirely transparent in the framework of twistor string theory, it is far from obvious in the CHY formulation.

We will now prove that the CHY Pfaffian which enters the Yang-Mills and gravity CHY formulas indeed vanishes when evaluated on solutions of the ‘wrong’ degree by constructing the kernel of the corresponding CHY matrix<sup>4</sup>. While the main motivation is to provide a succinct proof of the refinement of the CHY Pfaffian, we will actually discover a shadow of the twistor string, and thus get a glimpse into the connection twistor string in 4d and the ambitwistor string in 10d.

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<sup>4</sup>A different proof of this was given in [56].

We recall the original definition of the CHY Pfaffian for YM/gravity by CHY [57] here for the reader's convenience: it is given in terms of the matrix

$$M = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \quad (1.46a)$$

whose  $n^{\times 2}$  sub-matrices are defined as

$$\begin{aligned} A_{ij} &= p_i \cdot p_j S(z_i, z_j) & \text{for } i \neq j, & & A_{ii} &= 0, \\ B_{ij} &= \varepsilon_i \cdot \varepsilon_j S(z_i, z_j) & \text{for } i \neq j, & & B_{ii} &= 0, \\ C_{ij} &= \varepsilon_i \cdot p_j S(z_i, z_j) & \text{for } i \neq j, & & C_{ii} &= -\varepsilon_i \cdot P(z_i). \end{aligned} \quad (1.46b)$$

Notice that this is the limit of eq. (1.38) where the locations of the picture changing operators coincide with those of the vertex operators. In this representation it becomes natural to think of  $M$  as being a  $(2n)^{\times 2}$  matrix, of co-rank two, and the half-integrand is then given by its canonically defined *reduced* Pfaffian. We will construct the kernel of this  $(2n)^{\times 2}$  matrix (1.46) here.

Consider an amplitude with  $|p^-|$  negative particles, and pick a solution to the scattering equations of degree  $d$ . Let us denote  $\Delta := |p^-| - 1 - d$  the discrepancy between external helicity configuration and the degree. Now, the CHY matrix eq. (1.46) has a kernel if  $\Delta \neq 0$ .

In fact, if  $0 < \Delta$ , then define

$$v_i = \gamma(z_i) t_i^{-1} \frac{[\xi_i | \text{Res}_{z_i} \tilde{\zeta}]}{[\xi_i \tilde{\lambda}_i]}, \quad w_i = -\gamma(z_i) t_i^{-1} [\tilde{\lambda}_i | \text{Res}_{z_i} \tilde{\zeta}] \quad (1.47)$$

for  $i = 1, \dots, n$ , where  $\gamma \in H^0(T^{1/2})$  is any holomorphic section of  $T^{1/2}$  and

$$\tilde{\zeta}^{\dot{\alpha}} \in H^0 \left( \mathcal{O}(-d) \otimes K \left[ \sum_{i=1}^n z_i \right] \right) \quad (1.48)$$

with the requirement that

$$\text{Res}_{z_i} \zeta = t_i \tilde{\lambda}_i \quad \forall i \in p^+. \quad (1.49)$$

Here  $\xi_i$  are the auxiliary spinors that enter the definition of the polarisation vectors when  $i \in p^-$ , and arbitrary spinors when  $i \in p^+$ . (Note that this requirement implies some simplifications of the kernel, e.g.  $w_i = 0$  for  $i \in p^+$ . It also implies that under a gauge transformation  $\varepsilon_i \rightarrow \varepsilon_i + p_i$  the kernel transforms as  $v_i \rightarrow v_i - w_i$ , which is

necessary for the following equation to be gauge covariant.) With these definitions a straightforward, though somewhat tedious, calculation shows that

$$\begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = 0 \quad (1.50)$$

on the support of the scattering equations. Counting the free parameters in  $\zeta$  and  $\gamma$  we find that the kernel is of dimension  $2\Delta + 2$ .

If  $\Delta < 0$  we may simply take the parity conjugate of the above construction, and thus we find that the kernel is of dimension  $2|\Delta| + 2$ .

While it is pleasant to have an explicit expression for the kernel of the CHY matrix when  $\Delta \neq 0$ , the real reward of this analysis is that we recognise the parameter space of the kernel precisely as the kernel of the dual Hodges matrix [3], i.e.

$$H^0 \left( \mathcal{O}(-d) \otimes K^{1/2} \left[ \sum_{i=1}^n z_i \right] \right), \quad (1.51)$$

after a suitable lift to account for the gauge spinor indices. Recall that when  $0 < \Delta$  the dual Hodges matrix has a kernel, while for  $\Delta < 0$  the Hodges matrix has a kernel. This is the first glimpse of the connection between the fermionic system which gives rise to the CHY Pfaffian [13], and the one that gives rise to the Hodges matrices [4, 5, 7]. It relates the world-sheet partition function with sources on both sides and shows that field-space on which the kinetic operator  $\bar{\partial}$  acts on both sides is essentially the same. Motivated by this discovery, and without trying to make it more precise, we will now explore this connection in more detail by constructing an explicit map between the correlators of these systems.

### 1.3.1 Chiral Splitting of Fermion Correlators

The four-dimensional twistor string formulas are written in terms of spinor helicity variables, which exploit the fact that the tangent bundle of Minkowski space splits as

$$T\mathbb{M} \simeq \mathbb{S}^+ \otimes \mathbb{S}^- \quad (1.52)$$

into the left and right-handed spin bundles. This is particularly powerful because the states of a quantum field theory fall into representations of the Little group, which has a rather simple action on the elements in the product bundle.

The factorisation of the tangent bundle echoes in the natural splitting of 4d twistor string formulas into left- and right- handed factors, which we refer to as *chiral splitting* here. Chiral splitting can be stated as the splitting of fermion correlators of the form

$$\left\langle \prod_i (\lambda_i \tilde{\lambda}_i) \cdot \psi(z_i) \right\rangle, \quad S = \int_{\mathbb{CP}^1} \eta_{\mu\nu} \psi^\mu(z) \bar{\partial} \psi^\nu(z), \quad (1.53)$$

where  $(\lambda_i \tilde{\lambda}_i)_\mu$  is a  $4d$  null vector in spinor helicity notation and  $\psi^\mu(z)$  is a left-moving fermionic spinor on the Riemann sphere, into two factors, each only involving the left handed  $\lambda_i$  and right handed  $\tilde{\lambda}_i$  respectively.

While chiral splitting is a general property of the correlators (1.53), the present work is interested in this because of their role in the CHY formulae. Indeed, all kinematic Pfaffians appearing in the CHY arise as correlators of this type. This means that the chiral splitting of the worldsheet correlators eq. (1.53) on the sphere lifts, via the scattering equations, to a chiral splitting of  $4d$  quantum field theory amplitudes. Thus, the only source for mixing left- and right-handed variables are the scattering equations themselves, which are universal and independent of which  $4d$  QFT we study.

The key step in the translation of the CHY integrands into spinor–helicity language is the factorisation of the kinematic Pfaffians into Hodges matrices [5, 3]. In this section we will describe chiral splitting purely in terms of generic fermion correlators on the sphere, and remain completely oblivious to the role these ingredients play in (ambi-) twistor scattering amplitudes. In other words, this section is solely concerned with the following mathematical statement:

Take  $2n$  points on the sphere  $z_i$  and to each point associate one un-dotted (left-handed) spinor  $\lambda_i$  and one dotted (right-handed) spinor  $\tilde{\lambda}_i$ . Then the following identity holds

$$\frac{\text{Pf} \left( \frac{\langle \lambda_i \lambda_j \rangle [\tilde{\lambda}_i \tilde{\lambda}_j]}{z_i - z_j} \right)^{i,j=1,\dots,2n}}{V(\{1, \dots, 2n\})} = \frac{\det \left( \frac{\langle \lambda_i \lambda_j \rangle}{z_i - z_j} \right)_{j \in b^c}^{i \in b}}{V(b) V(b^c)} \frac{\det \left( \frac{[\tilde{\lambda}_i \tilde{\lambda}_j]}{z_i - z_j} \right)_{j \in \tilde{b}^c}^{i \in \tilde{b}}}{V(\tilde{b}) V(\tilde{b}^c)} \quad (1.54)$$

where  $b, \tilde{b}$  are arbitrary ordered<sup>5</sup> subsets of  $\{1, \dots, 2n\}$  of size  $n$  and  $b^c, \tilde{b}^c$  are their complements. We use the notation that  $\det(M_{ij})_{j \in b}^{i \in a}$  denotes the determinant of the matrix  $M$ , with rows indexed by the set  $a$  and columns by the set  $b$ . Since the Pfaffian is only defined for antisymmetric matrices, it's rows and columns are necessarily indexed

<sup>5</sup>The expression eq. (1.54) is easily seen to be independent of the ordering of  $b, \tilde{b}$ , but the Hodges determinant and the Vandermonde determinant separately are not, so we keep track of the ordering.



by the same set. We also use the Vandermonde determinant, defined as usual

$$V(b) := \prod_{i < j \in b} \frac{1}{S(z_i, z_j)} \simeq \prod_{i < j \in b} (z_i - z_j) \quad (1.55)$$

for an ordered set of points on the sphere, and we recall that  $S(z_i, z_j)$  is the genus zero Szegő kernel

$$S(z_i, z_j) \equiv \frac{\sqrt{dz_i dz_j}}{z_i - z_j} \simeq \frac{1}{z_i - z_j}.$$

The last equality uses the isomorphism  $K^{1/2} \simeq \mathcal{O}(-1)$  on the sphere.

It is worth emphasizing that the factorisation eq. (1.54) does not require the scattering equations to hold (and that the spinors  $\lambda_i, \tilde{\lambda}_i$  do a priori not have any interpretation in terms of null momenta or polarisation vectors). The kinematic Pfaffians in the CHY formulations for gravity, EYM, etc. may all be realised as appropriate limits or special cases of this Pfaffian. An example of this follows after the proof.

### Proof of Chiral Splitting

To prove eq. (1.54) we simply compute the residues as any  $z_i - z_j \rightarrow 0$  on both sides and invoke induction. At first glance it seems as though the right hand side depends on the splitting of the  $2n$  points into the two halves  $b, b^c$  and  $\tilde{b}, \tilde{b}^c$  respectively, which would be at odds with the manifest  $S_{2n}$  antisymmetry of the Pfaffian on the left. This tension is resolved by the surprising fact that the combination

$$\frac{\det \left( \frac{\langle \lambda_i \lambda_j \rangle}{z_i - z_j} \right)_{j \in b^c}^{i \in b}}{V(b) V(b^c)} \quad (1.56)$$

is totally  $S_{2n}$  permutation symmetric, despite making only the permutation invariance under a  $S_n \times S_n \times \mathbb{Z}_2$  subgroup manifest. To exhibit full permutation invariance we may go to an alternative representation

$$\frac{\det \left( \frac{\langle \lambda_i \lambda_j \rangle}{z_i - z_j} \right)_{j \in b^c}^{i \in b}}{V(b) V(b^c)} = (-1)^{\frac{n(n-1)}{2}} \sum_{\substack{p \subset \{1, \dots, 2n\} \\ |p|=n}} \frac{\prod_{i \in p} (\lambda_i)^0 \prod_{j \in p^c} (\lambda_j)^1}{\prod_{\substack{i \in p \\ j \in p^c}} (z_i - z_j)} \quad (1.57)$$

where the sum runs over all unordered subsets  $p \subset \{1, \dots, 2n\}$  of size  $n$ . The right hand side is now manifestly  $S_{2n}$  permutation invariant (though it has lost its manifest

$SL(2)$  Lorentz invariance). This last equality is interesting in its own right, and we reflect on it below.

### Proof of $S_{2n}$ Symmetry

We begin by establishing the full  $S_{2n}$  symmetry of eq. (1.56), by proving the relation eq. (1.57). For simplicity, and without loss of generality, we assign the labels  $\{1, \dots, n\}$  to the rows and  $\{n+1, \dots, 2n\}$  to the columns, so we now want to prove the  $S_{2n}$  permutation symmetry of

$$\frac{\det \left( \Phi_{j=n+1, \dots, 2n}^{i=1, \dots, n} \right)}{V(\{1, \dots, n\}) V(\{n+1, \dots, 2n\})} . \quad (1.58)$$

The plan is to examine the poles and residues on both sides of eq. (1.57) as any of the two punctures coincide, and then use a recursion argument to show that the residues agree. First, we rewrite the claim as

$$\det \left( \Phi_{j=n+1, \dots, 2n}^{i=1, \dots, n} \right) = (-1)^{\frac{n(n-1)}{2}} \sum_{\substack{\mathbf{b} \subset \{1, \dots, 2n\} \\ |\mathbf{b}|=n}} \prod_{i \in \mathbf{b}} (\lambda_i)^0 \prod_{j \in \mathbf{b}^c} (\lambda_j)^1 \frac{V(1, \dots, n) V(n+1, \dots, 2n)}{\prod_{\substack{i \in \mathbf{b} \\ j \in \mathbf{b}^c}} (z_i - z_j)} , \quad (1.59)$$

Each side is now a section of  $\otimes_i \mathcal{O}_i(-1)$  with at most simple poles as any two punctures coincide, so by Cauchy's theorem, comparing residues is sufficient to prove equality. Furthermore, given the already manifest  $S_n \times S_n \times \mathbb{Z}_2$  symmetry, it is sufficient to check the residues at  $z_1 = z_2$  and  $z_1 = z_{2n}$ . It is actually immediately clear that both sides have vanishing residue at  $z_1 = z_2$ , so we only have to put some effort into checking the residue at  $z_1 = z_{2n}$ . On the left hand side we find

$$\text{Res}_{z_1=z_{2n}} \left\{ \det \left( \Phi_{j=n+1, \dots, 2n}^{i=1, \dots, n} \right) \right\} = (-1)^n \langle \lambda_1, \lambda_{2n} \rangle \det \left( \Phi_{j=n+1, \dots, 2n-1}^{i=2, \dots, n} \right) , \quad (1.60)$$

while on the right hand side we find

$$\begin{aligned} & \text{Res}_{z_1=z_{2n}} \left\{ \sum_{\substack{\mathbf{b} \subset \{1, \dots, 2n\} \\ |\mathbf{b}|=n}} \prod_{i \in \mathbf{b}} (\lambda_i)^0 \prod_{j \in \mathbf{b}^c} (\lambda_j)^1 \frac{V(1, \dots, n) V(n+1, \dots, 2n)}{\prod_{\substack{i \in \mathbf{b} \\ j \in \mathbf{b}^c}} (z_i - z_j)} \right\} \\ &= \langle \lambda_{2n}, \lambda_1 \rangle \sum_{\substack{\mathbf{b} \subset \{2, \dots, 2n-1\} \\ |\mathbf{b}|=n-1}} \prod_{i \in \mathbf{b}} (\lambda_i)^0 \prod_{j \in \mathbf{b}^c} (\lambda_j)^1 \frac{V(2, \dots, n) V(n+1, \dots, 2n-1)}{\prod_{\substack{i \in \mathbf{b} \\ j \in \mathbf{b}^c}} (z_i - z_j)} \end{aligned} \quad (1.61)$$

where, going to the second line, we observed that only those terms in the sum where 1 and  $2n$  are in different subsets contribute to the pole. We immediately recognise the condition for the residues to agree as the very same claim we're trying to prove, but for  $n - 1$ . Hence, we may conclude the proof by invoking a simple induction argument from  $n$  to  $n - 1$ .

### Proof of Splitting *cont.*

Armed with the knowledge that the factor eq. (1.56) is secretly  $S_{2n}$  symmetric, we may now establish the factorisation formula eq. (1.54) by comparing residues. Both sides are again sections of  $\otimes_i \mathcal{O}_i(-1)$  so comparing residues as any pair of punctures collide is sufficient to prove equality.

Using the  $S_{2n}$  symmetry of both sides we may simply look at the residue as  $z_1 \rightarrow z_2$ , where we find for the left hand side

$$\text{Res}_{z_1 \rightarrow z_2} \left\{ \text{Pf} \left( \frac{\langle i j \rangle [i j]}{z_i - z_j} \right)^{i,j=1,\dots,2n} \right\} = \langle \lambda_1 \lambda_2 \rangle [\tilde{\lambda}_1 \tilde{\lambda}_2] \text{Pf} \left( \frac{\langle i j \rangle [i j]}{z_i - z_j} \right)^{i,j=3,\dots,2n}. \quad (1.62)$$

On the right hand side we first make a judicious choice for the splitting of labels into rows and columns such that the rows of the first matrix be labelled by the set  $\{1\} \cup b'$  and the columns by  $\{2\} \cup b'^c$  where  $b' \cup b'^c = \{3, \dots, 2n\}$  is a partition of the remaining labels and similarly for the second matrix. (For the sake of clarity we drop the primes below.) Hence we find the residue on the right hand side

$$\begin{aligned} \text{Res}_{z_1 \rightarrow z_2} & \left\{ \frac{\det \left( \frac{\langle i j \rangle}{z_i - z_j} \right)_{j \in \{2\} \cup b^c}^{i \in \{1\} \cup b}}{V(\{1\} \cup b) V(\{2\} \cup b^c)} \frac{\det \left( \frac{[i j]}{z_i - z_j} \right)_{j \in \{2\} \cup \tilde{b}^c}^{i \in \{1\} \cup \tilde{b}}}{V(\{1\} \cup \tilde{b}) V(\{2\} \cup \tilde{b}^c)} V(\{1, \dots, 2n\}) \right\} \\ &= \langle \lambda_1 \lambda_2 \rangle [\tilde{\lambda}_1 \tilde{\lambda}_2] \frac{\det \left( \frac{\langle i j \rangle}{z_i - z_j} \right)_{j \in b^c}^{i \in b}}{V(b) V(b^c)} \frac{\det \left( \frac{[i j]}{z_i - z_j} \right)_{j \in \tilde{b}^c}^{i \in \tilde{b}}}{V(\tilde{b}) V(\tilde{b}^c)} V(\{3, \dots, 2n\}) \end{aligned} \quad (1.63)$$

Notice that each determinant has a simple pole as  $z_1 \rightarrow z_2$ , while the big Vandermonde factor has a simple zero, and on the location of the residue there are cancellations between the various Vandermonde factors. We again recognise the condition for the residues to agree as the very same claim we're trying to prove but for  $n - 1$  so, after invoking recursion, this concludes the proof of the splitting formula eq. (1.54).

### 1.3.2 A CFT Perspective

The rational functions given in terms of Pfaffians and determinants that enter eq. (1.54) have natural origins in  $2d$  CFT on the punctured Riemann sphere. We believe that  $2d$  CFT is the proper realm for understanding eq. (1.54) and in fact the crucial equality eq. (1.57) was discovered using that CFT description. Here we briefly sketch this relation but leave the details for an upcoming publication.

The left hand side of eq. (1.54) is given by the correlator

$$\left\langle \prod_{i=1}^{2n} (\lambda_i \tilde{\lambda}_i) \cdot \psi(z_i) \right\rangle = \text{Pf} \left( \langle \lambda_i \lambda_j \rangle [\tilde{\lambda}_i \tilde{\lambda}_j] S(z_i, z_j) \right)^{i,j=1,\dots,2n} \quad (1.64)$$

with the action

$$S = \int_{\mathbb{P}^1} \psi^\mu \bar{\partial} \psi^\nu \eta_{\mu\nu} , \quad \psi \in \Pi\Omega^0(K^{1/2} \otimes T\mathbb{M}) , \quad (1.65)$$

and no constraints on the locations  $z_i$ . On flat Minkowski space  $\mathbb{M} = \mathbb{R}^{3,1}$  the tangent bundle splits into a product of the left-handed and right-handed spin bundles

$$T\mathbb{M} \simeq \mathbb{S}^+ \otimes \mathbb{S}^- , \quad (1.66)$$

where the isomorphism is provided by the van der Waerden symbols  $\sigma_{\alpha\dot{\alpha}}^\mu$ . We can write the world-sheet current<sup>6</sup> associated to Lorentz transformations on either side of the isomorphism as

$$\psi^{[\mu} \psi^{\nu]} \simeq \rho_\alpha^a \rho_\beta^b \varepsilon_{ab} \varepsilon_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\alpha\beta} \tilde{\rho}_{a\dot{\alpha}} \tilde{\rho}_{b\dot{\beta}} \varepsilon^{ab} \quad (1.67)$$

with the new fields

$$\rho^a \in \Pi\Omega^0(K^{1/2}) \otimes \mathbb{S}^- , \quad \tilde{\rho}_a \in \Pi\Omega^0(K^{1/2}) \otimes \mathbb{S}^+ , \quad (1.68)$$

where the Roman indices  $a, b = 1, 2$  label the fundamental representation of a new  $\text{SL}(2)$  gauge symmetry. It is essentially the little group of a massive particle, and can be seen to arise as redundancy in the change of variables from  $\psi$  to  $\rho, \tilde{\rho}$ . We will see that is responsible for the permutation symmetry of (1.56).

While the isomorphism eq. (1.67) is straightforward at the level of the currents, the equality eq. (1.54) suggests that the corresponding  $2d$  sigma models are in some sense equivalent.

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<sup>6</sup>The argument in [56] is based on this identity.

Indeed, the right hand side of eq. (1.54) involves the correlators

$$\left\langle \prod_{i \in \mathbf{b}} \langle \lambda_i, \rho^1(z_i) \rangle \prod_{j \in \mathbf{b}^c} \langle \lambda_j, \rho^2(z_j) \rangle \right\rangle = \det \left( \langle \lambda_i \lambda_j \rangle S(z_i, z_j) \right)_{i \in \mathbf{b}, j \in \mathbf{b}^c}^{i \in \mathbf{b}} \quad (1.69)$$

in the action

$$S = \int_{\mathbb{P}^1} \varepsilon_{ab} \langle \rho^a, \bar{\partial} \rho^b \rangle, \quad (1.70)$$

and likewise for the right-handed determinant. The complete building block eq. (1.56) can be computed as a correlator in the CFT

$$S = \int_{\mathbb{P}^1} \langle \mu, \bar{\partial} \lambda \rangle + \varepsilon_{ab} \langle \rho^a, \bar{\partial} \rho^b \rangle + \chi_a \langle \mu, \rho^a \rangle, \quad (1.71)$$

with the new bosonic fields

$$\lambda \in \Omega^0(K^{1/2} \otimes \mathcal{O}(n-1)) \otimes \mathbb{S}^- , \quad \mu \in \Omega^0(K^{1/2} \otimes \mathcal{O}(1-n)) \otimes \mathbb{S}^- . \quad (1.72)$$

The last term contains the Lagrange multiplier  $\chi^a$  and means that the current  $\langle \mu, \rho_a \rangle$  is gauged. In this model, the expression eq. (1.56) is given by the correlator

$$\left\langle \delta^{(2n-2)}(\lambda(z_*)) \prod_{i=1}^{2n} \delta^2(\gamma(z_i)) \langle \lambda_i, \lambda(z_i) \rangle \right\rangle \quad (1.73)$$

where the delta function  $\delta^{(2n-2)}(\lambda(z_*))$  is inserted to kill the zero modes of  $\lambda(z)$  and the location  $z_*$  drops out of the correlator. At this stage, the full  $S_{2n}$  permutation symmetry is completely manifest. Following standard descent procedure along the orbits generated by the current  $\langle \mu \rho_a \rangle$ , we have to choose which vertex operators to descend by  $\langle \mu \rho_1 \rangle$  and  $\langle \mu \rho_2 \rangle$ , and thus break  $S_{2n}$  symmetry, yielding the various representations

$$\left\langle \prod_{i \in \mathbf{b}} \delta(\gamma^1(z_i)) \langle \lambda_i, \rho^1(z_i) \rangle \prod_{j \in \mathbf{b}^c} \delta(\gamma^2(z_j)) \langle \lambda_j, \rho^2(z_j) \rangle \right\rangle \quad (1.74)$$

where the explicitly shown superscripts 1, 2 are  $SL(2)$  indices.

Having realised the building block eq. (1.56) as a CFT correlator, we observe that the Moebius weight of the vertex operators in eq. (1.73) is left invariant by a “twist” of the Moebius weights of the fermionic fields  $\rho^a$  by a  $U(1)$  subgroup of the  $SL(2)$ . More explicitly, we can pick some  $U(1)$  subgroup, change the notation for the fermionic fields to  $(\rho^1, \rho^2) \rightarrow (\rho, \bar{\rho})$ , so that the action becomes that of a single complex fermion, and

assign the Moebius weights

$$\rho \in \Pi\Omega^0(K^{1/2} \otimes \mathcal{O}(\delta)) \otimes \mathbb{S}^- , \quad \bar{\rho} \in \Pi\Omega^0(K^{1/2} \otimes \mathcal{O}(-\delta)) \otimes \mathbb{S}^- , \quad (1.75)$$

with an arbitrary integer  $\delta$ . Performing the analogous changes on the Lagrange multiplier  $\chi$  and the corresponding Faddeev-Popov ghosts, we find that the action

$$S = \int_{\mathbb{P}^1} \langle \mu, \bar{\partial}\lambda \rangle + \langle \bar{\rho}, \bar{\partial}\rho \rangle + \chi \langle \mu, \bar{\rho} \rangle + \bar{\chi} \langle \mu, \rho \rangle , \quad (1.76)$$

and the correlator

$$\left\langle \delta^{(2n-2)}(\lambda(z_*)) \prod_{i=1}^{2n} \delta^2(\gamma(z_i)) \langle \lambda_i, \lambda(z_i) \rangle \right\rangle \quad (1.77)$$

remain unchanged in form, and still make sense. In fact they, still compute exactly the same object as eq. (1.73) before the “twisting”, i.e. the correlator eq. (1.77) is independent of  $\delta$ . Indeed, setting  $\delta = n - 1$  we find that eq. (1.77) evaluates to the right hand side of eq. (1.57), and in fact this is how the author found the equality (1.57). This interpretation was of course not necessary for the proof, and the statement that the correlator (1.77) is indeed independent of  $\delta$  can be carried out without any knowledge or regard for the CFT origin, in analogy to the prove above, by comparing residues and invoking recursion.

### 1.3.3 Chiral Splitting for Gravity

We would like to point out that the chiral factorisation formula (1.54) does immediately apply to the CHY Pfaffian for gravity of [9] (eq. (1.46)), essentially since the diagonal entries of the off-diagonal sub-matrix  $C$  represent an obstruction to writing

$$M_{ab} = q_a \cdot q_b S(w_a, w_b) , \quad (1.78)$$

for  $a, b, = 1, \dots, 2n$ . To more more explicit, we would like to set  $q_a = p_i$  when  $a = 1, \dots, n$  and  $q_a = \varepsilon_{i-n}$  when  $a = n + 1, \dots, 2n$ , while while for the locations we take  $w_a = z_{i \bmod n}$ . It is worth emphasizing that this is not a fatal obstruction to applying eq. (1.54), as there are several ways to overcome this.

The most natural way to bring the CHY Pfaffian into a form amenable to the chiral factorisation formula (1.54) is to keep the picture changing operators at locations

distinct from the vertex operators, i.e. to consider the correlator (1.37a)

$$\left\langle \prod_{i=1}^n \varepsilon \cdot \psi(z_i) \prod_{r=1}^{n-2} \psi(x_r) \cdot P(x_r) \right\rangle . \quad (1.79)$$

In addition to that, we use the parametrisation

$$\varepsilon_i^- = \frac{|\lambda_i\rangle [\xi_i]}{[\xi_i \tilde{\lambda}_i]} , \quad \varepsilon_i^+ = \frac{|\xi_i\rangle [\tilde{\lambda}_i]}{\langle \xi_i \lambda_i \rangle} , \quad (1.80)$$

of the polarisation vectors in terms of arbitrary auxiliary spinors  $\xi, \tilde{\xi}$ , as well as writing

$$P(z) = |\lambda(z)\rangle [\tilde{\lambda}(z)] \quad (1.81)$$

which holds on the support of the scattering equations.

After applying the chiral factorisation formula (1.54) to this Pfaffian, we may again take the limit where the PCOs coincide with some of the vertex operators. After a few simple steps using linearity and antisymmetry of the determinant [33] we recover the original Hodges matrices [3].

We remark that the factorisation of  $P(z)$  as a rank one bi-spinor is the only step in the chiral factorisation of the CHY Pfaffian which requires the scattering equations; the splitting of a Pfaffian into two determinants holds even without the scattering equations.

### 1.3.4 Discussion

It is very tempting to apply eq. (1.54) to scattering-equation based formulas for higher-loop amplitudes, which currently come in two flavours. On the one hand, the ambitwistor string model [13] gives rise to amplitudes for supergravity on the torus [26]. Indeed, eq. (1.54) has a natural generalisation to higher genus surfaces, and we expect a generalisation of the proof here to carry over. It is however believed that the ambitwistor string is only modular invariant in  $10d$ , so even though the external states can easily be restricted to lie in a 4d subspace, the loop momentum would have to be integrated over a  $10d$  space, which will make  $P(z)$  generically a  $10d$  vector. This obstructs the use of eq. (1.54) as shown here since the  $C_{ii}$  elements of the CHY type Pfaffian cannot be split straightforwardly. Further complications might arise from Ramond sector fields or the spin-structure dependence of the Szegő kernel.

On the other hand, there are formulas for loop amplitudes on the nodal sphere [18–20]. While these seem to be well defined (or at least come with a canonical regularisation scheme) in any dimension, the above obstruction remains: on the nodal sphere the scattering equations imply generically  $P(z)^2 \neq 0$ , so we again cannot split the  $C_{ii}$  elements of the CHY type Pfaffian straightforwardly. In this situation the resolution might be more apparent: we can write  $P(z)$  as a sum of null vectors, and, using the multi-linearity of the Pfaffian, apply the factorisation eq. (1.54) to each summand separately. There have just been promising new results [58] for  $4d$  loop amplitudes based on the  $4d$  refinement of the scattering equations on the nodal sphere, which might be combined naturally with the present work to find  $n$ -point SUGRA integrands. We leave these exciting thoughts and questions for future work.

## 1.4 4d Einstein-Yang-Mills in Twistor Space

In this section, we present a world-sheet formula for all tree-level scattering amplitudes, in all trace sectors, of four-dimensional  $\mathcal{N} \leq 4$  supersymmetric Einstein-Yang-Mills theory, based on the refined scattering equations. This formula generalises previously known formulas for all-trace purely bosonic, or supersymmetric single-trace amplitudes.

Following the early work [59–61] there has been renewed interest recently in the study of Einstein-Yang-Mills amplitudes and their relation to pure Yang-Mills from the perspective of the double copy construction [62, 63], string theory [64, 65] and the CHY formulae [66–69]. One motivation for the present work is that the new formulas for  $\mathcal{N} = 4$  EYM scattering amplitudes (eqs. (1.88) and (1.96)) can provide a new tool to study these relations, particularly in light of the  $4d$  KLT and BCJ relations [70].

The formula we propose is a marriage of the RSVW [1, 53] formula for  $\mathcal{N} = 4$  super-Yang-Mills

$$\sum_d \int_{\mathcal{M}_{0,n}(d)} d\mu_d \text{ PT} \prod_{i \in g} A_i(Z(z_i)) \quad (1.82)$$

and the CS formula [3, 71] for  $\mathcal{N} = 8$  super-gravity

$$\sum_d \int_{\mathcal{M}_{0,n}(d)} d\mu_d \det'(\Phi) \det'(\tilde{\Phi}) \prod_{i \in h} h_i(Z(z_i)) \quad (1.83)$$

The main ingredients are the world-sheet Parke-Taylor factor  $\text{PT}$  and the reduced determinants of the Hodges and dual Hodges matrices. We will recall the ingredients of these formulae below.



We start by recalling some basic facts and constraints on the form of tree-level scattering amplitudes in EYM. The most immediate constraints are that any proposal for an EYM amplitude has to reduce to the known formulas for  $\mathcal{N} = 4$  Yang-Mills [53, 1] and  $\mathcal{N} = 8$  gravity [3] suitably reduced to  $\mathcal{N} = 4$ .

We consider  $\mathcal{N} = 4$  EYM with one adjoint valued vector multiplet and two parity conjugate graviton multiplets. Just like the amplitudes of sYM and super-gravity, those of  $\mathcal{N} = 4$  EYM are organised by  $R$  charge sector. When restricting the external states to gluons and gravitons, this means that amplitudes are supported on solutions to the scattering equations of degree  $d = n_{gr}^- + n_{gl}^- - 1$ . This can already be seen from the CHY representation [72, 10] of the bosonic EYM amplitudes: The CHY integrand for EYM still contains one vector mode Pfaffian, in which gravitons and gluons enter in the same way, and have shown above that this Pfaffian vanishes when evaluated on solutions of the wrong degree.

A further constraint comes from the spacetime Lagrangian: it dictates that a tree level scattering amplitude in Einstein-Yang-Mills in the  $\tau$  trace sector comes with a factor

$$\kappa^{n_{gr}+2\tau-2} \quad (1.84)$$

of the gravitational coupling constant  $\kappa \sim \sqrt{G_N}$ , where  $n_{gr}$  denotes the number of external gravitons. In [3] it was explained that, when written in terms of a worldsheet model, these powers of  $\kappa$  must be accompanied by the same number of powers of  $\langle, \rangle$  or  $[, ]$  brackets. Indeed, from dimensional analysis we find that

$$\#\langle, \rangle + \#[, ] = n_{gr}^+ + n_{gr}^- + 2\tau - 2. \quad (1.85)$$

Parity conjugation exchanges  $\langle, \rangle$  and  $[, ]$ , which fixes

$$\#\langle, \rangle = n_{gr}^- + \tau - 1, \quad \#[, ] = n_{gr}^+ + \tau - 1. \quad (1.86)$$

From the perspective of twistor theory, the appearance of the  $SL(2)_{L,R}$  invariants  $\langle, \rangle$  and  $[, ]$  controls the breaking of conformal symmetry of a theory, and the very existence of a well defined counting is a hallmark of the natural action (and breaking) of this symmetry on twistor space.

A less obvious constraint arises by considering the trace sector of EYM amplitudes with exactly two gluons per trace. This part of the amplitude can be computed from the  $\mathcal{N} = 8$  formula for super-gravity, by assigning different  $R$ -charge values to pairs of vector modes. It is equivalent to the Einstein-Maxwell amplitude with  $n_{tr}$  flavours of

photons. Strictly speaking, this equivalence only holds for a sufficiently low number of traces  $n_{tr}$ , since there is only a finite number of distinct  $R$ -charge combinations we can use to distinguish the pairs of gluons, but at tree level, this constraint turns out to be irrelevant.

Considering all these constraints, we can construct a proposal for ‘twistor-string’ formula for  $\mathcal{N} = 4$  super Einstein-Yang-Mills

## $\mathcal{N} = 4$ sEYM on Twistor Space

In general, a scattering amplitude is a multi-linear functional of the external wave functions. Most commonly it is simply given in a basis of plane waves, but on twistor space it is actually more natural to keep the full structure. Using  $\mathcal{N} = 4$  onshell SUSY we may write the wave function for a whole SUSY multiplet as a single function on onshell superspace. In  $\mathcal{N} = 4$  sEYM there are two colour neutral multiplets,  $h_i, \phi_i$ , which contain the graviton as their highest/lowest spin state, and one adjoint-valued multiplet  $A_i$ , containing the gluons. Via the Penrose transform the external wave functions of the super-multiplets are given by cohomology classes with a certain homogeneity on super twistor space  $\mathbb{PT} := \mathbb{CP}^{3|4} \setminus \mathbb{CP}^{1|4}$

$$h_i \in H^1(\mathbb{PT}, \mathcal{O}(2)), \quad A_i \in H^1(\mathbb{PT}, \mathcal{O}(0)), \quad \phi_i \in H^1(\mathbb{PT}, \mathcal{O}(-2)), \quad (1.87)$$

of helicity  $-2, -1, 0$  respectively. As usual, the coefficients in the Taylor expansion w.r.t. the Grassmann coordinates of  $\mathbb{PT}$  are the various components of the supermultiplet. With these definitions in place, the sEYM scattering amplitude in the  $n_{tr} = \tau$  colour trace sector on twistor space is

$$\sum_d \int_{\mathcal{M}_{0,n}(d)} d\mu_d \sum_{\substack{i_2 < j_2 \in \text{tr}_2 \\ \dots \\ i_m < j_m \in \text{tr}_m}} \frac{\det(\Phi_{j \in \phi \cup J}^{i \in \phi \cup I})}{V(I) V(J)} \frac{\det(\tilde{\Phi}_{j \in h \cup J}^{i \in h \cup I})}{V(I) V(J)} \frac{V(I \cup J)}{\prod_{\alpha=2}^{\tau} S(z_{i_\alpha}, z_{j_\alpha})} \prod_{\alpha=1}^{\tau} \text{PT}(\text{tr}_\alpha) \prod_{i \in h} h_i(Z(z_i)) \prod_{i \in g} A_i(Z(z_i)) \prod_{i \in \phi} \phi_i(Z(z_i)) \quad (1.88)$$

where  $V(\cdot)$  denotes the Vandermode determinant (1.55) and we abbreviated the sets

$$I \equiv \{i_2, \dots, i_\tau\} \quad \text{and} \quad J \equiv \{j_2, \dots, j_\tau\}, \quad (1.89)$$

as well as the measure

$$d\mu_d \equiv \frac{d^{4(d+1)|4(d+1)} Z}{\text{vol GL}(2, \mathbb{C})} \equiv \frac{\prod_{a=0}^d d^{4|4} Z_a}{\text{vol GL}(2, \mathbb{C})} \quad (1.90)$$

on  $\mathcal{M}_{0,n}(d)$ , the moduli space of holomorphic maps of degree  $d$  from the  $n$ -punctured Riemann sphere to super twistor space. For the last term we have chosen to coordinatise this space as  $Z(z) = \sum_{a=0}^d Z_a s_a(z)$ , for some fixed basis<sup>7</sup> of polynomials  $\{s_a\}_{a=0}^d$  spanning  $H^0(\mathbb{P}^1, \mathcal{O}(d))$ . The Hodges matrices  $\Phi, \tilde{\Phi}$  are defined on twistor space by

$$\Phi_{ij} = S(z_i, z_j) \langle Z(z_i) Z(z_j) \rangle, \quad \Phi_{ii} = \langle Z(z_i) dZ(z_i) \rangle \quad (1.91)$$

and

$$\tilde{\Phi}_{ij} = S(z_i, z_j) \left[ \frac{\partial}{\partial Z(z_i)} \frac{\partial}{\partial Z(z_j)} \right], \quad \tilde{\Phi}_{ii} = - \sum_{j \neq i} \tilde{\Phi}_{ij} \frac{p(z_j)}{p(z_i)} \quad (1.92)$$

for some arbitrary section  $p \in H^0(T^{1/2} \otimes \mathcal{O}(d))$ . Here  $\langle \cdot, \cdot \rangle$  and  $[\cdot, \cdot]$  are the infinity twistor<sup>8</sup> and dual infinity twistor<sup>8</sup> respectively, i.e.  $\langle ZZ' \rangle = \mathcal{I}_{IJ} Z^I Z'^J$  for two twistors  $Z, Z' \in \mathbb{PT}$  and  $[WW'] = \tilde{\mathcal{I}}^{IJ} W_I W'_J$  for two dual twistors  $W, W' \in \mathbb{PT}^*$ . Generally, the appearance of the infinity twistor signals and controls the breaking of space-time conformal symmetry, which on twistor space is represented by general linear transformations. In the present case, they reduce as  $\langle ZZ' \rangle = \langle \lambda \lambda' \rangle$  and  $[WW'] = [\tilde{\lambda} \tilde{\lambda}']$  to the Lorentz invariant pairings of left- and right-handed spinors, respectively.

Furthermore, we used the familiar world-sheet Parke-Taylor factor of a gluon trace, defined as

$$\text{PT}(\text{tr}) := \sum_{\sigma \in S_{|\text{tr}|}/\mathbb{Z}_{|\text{tr}|}} \text{Tr} \left[ \mathbf{T}_{\sigma(1)} \cdots \mathbf{T}_{\sigma(|\text{tr}|)} \right] \prod_{i \in \text{tr}} S(z_{\sigma(i)}, z_{\sigma(i+1)}) , \quad (1.93)$$

with the gauge group generators  $\mathbf{T}_i$  associated to each gluon in the trace.

It is worth pointing out that while  $\text{tr}_1$  appears to be singled out in eq. (1.88), the scattering equations guarantee that the formula is independent of this choice, so is actually  $S_\tau$  permutation symmetric.

<sup>7</sup>Note that since  $\mathbb{PT}$  is a Calabi-Yau supermanifold, the holomorphic measure  $d\mu_d$  is independent of the choice of basis  $\{s_a\}_{a=0}^d$  by itself. Indeed, when the target space is  $\mathbb{CP}^{m|\mathcal{N}}$ , a change of basis in  $H^0(\mathbb{P}^1, \mathcal{O}(d))$  with Jacobian  $J(\{s_a\}, \{s'_a\})$  induces the integration measure to transform as  $d\mu_d \rightarrow d\mu_d J(\{s_a\}, \{s'_a\})^{m+1-\mathcal{N}}$ .

<sup>8</sup>These are fixed simple bivectors (antisymmetric matrices of rank 2) on twistor space, which arise in the decompactification of  $\mathbb{M}$  to  $\mathbb{M}$ .

The formula (1.88) reveals a striking feature of EYM scattering amplitudes: on twistor space, the dependence on the infinity twistor and dual infinity twistor has largely, but not entirely, separated. This is reminiscent of the separation in  $\mathcal{N} = 8$  supergravity amplitudes [3, 7], but the addition of Yang-Mills interactions leads to a sum of such products. In other words, the presence of gluon traces obstructs a complete separation of the infinity twistor and dual infinity twistor, albeit in a rather systematic way.

Another property of the amplitudes can be learned from eq. (1.88): since the Hodges determinant is an antisymmetric polynomial of degree  $d - 1$  in the marked points, it will vanish identically if  $d < |h| + \tau - 1$ . Hence we find, for non-vanishing amplitudes, the inequality  $d + 1 \geq |h| + \tau$ , and similarly the parity conjugate  $\tilde{d} + 1 \geq |\phi| + \tau$ . Moreover, since the  $R$ -charge selection rules follow from the fermionic part of the map and wave-functions, which completely separates from the rest of the formula, we manifestly have the usual selection rules for  $\mathcal{N} = 4$  SUSY (in particular  $k = d + 1$ ). This completely fixes the degree  $d$  in terms of the external states, e.g. for external gravitons and gluons only, we recover  $d + 1 = n_{gr}^- + n_{gl}^-$  as expected. A corollary of this is that  $n_{gl}^- \geq \tau$  and  $n_{gl}^+ \geq \tau$ , so any amplitude with less negative/positive gluons than traces will vanish.

It is worth pointing out that the manifest  $\mathcal{N} = 4$  space-time supersymmetry and the separation of left- and right-handed variables is unique to the twistor representation of the amplitude. Neither of these properties is obvious/accessible from simply using the substitution  $p_i \cdot p_i \rightarrow \langle \lambda_i \lambda_j \rangle [\lambda_i \lambda_j]$  in the dimension agnostic CHY formulae.

### Einstein-Yang-Mills amplitudes in 4d spinor helicity variables

We may easily go from the twistor space representation to the so-called ‘ambitwistor representation’ (1.96) [73] by specifying the external states to be plane wave states and use the explicit form of the Penrose representative

$$\int \frac{dt_i}{t_i} t_i^{2s_i+2} \bar{\delta}^2(\lambda_i - t_i \lambda) \exp\left(t_i [\tilde{\lambda}_i \mu] + t_i \eta_i \cdot \chi\right) \quad (1.94)$$

for a multiplet of helicity  $s_i$ . Indeed, by judiciously choosing a coordinate basis<sup>7</sup> for the space of maps that is adapted to the external data,

$$Z(z) = \sum_{i \in \mathbf{p}^-} Z_i \prod_{j \in \mathbf{p}^- \setminus \{i\}} \frac{z - z_j}{z_i - z_j} \quad (1.95)$$

while keeping the punctures fixed, we may perform the integral over the moduli space of the map  $Z(z)$  trivially.

Restricting external states to gluons and gravitons, we thus find the Einstein-Yang-Mills scattering amplitudes in the  $\tau$ -trace sector

$$\int \frac{1}{\text{vol GL}(2, \mathbb{C})} \sum_{\substack{i_2 < j_2 \in \text{tr}_2 \\ i_\tau < j_\tau \in \text{tr}_\tau}} \frac{\det(\Phi_{j \in h^- \cup J}^{i \in h^- \cup I})}{V(I) V(J)} \frac{\det(\tilde{\Phi}_{j \in h^+ \cup J}^{i \in h^+ \cup I})}{V(I) V(J)} \frac{V(I \cup J)}{\prod_{\alpha=2}^\tau S(z_{i_\alpha}, z_{j_\alpha})} \\ \prod_{\alpha=1}^\tau \text{PT}(\text{tr}_\alpha) \prod_{i \in p^-} \frac{d\tilde{u}_i}{\tilde{u}_i} \bar{\delta}^2(\tilde{\lambda}_i - \tilde{u}_i \tilde{\lambda}(z_i)) \prod_{i \in p^+} \frac{du_i}{u_i} \bar{\delta}^2(\lambda_i - u_i \lambda(z_i)) \quad (1.96)$$

where  $h^\pm$  denote the positive/negative helicity gravitons,  $p^\pm$  is set of all positive/negative helicity particles and we used again the abbreviations

$$I \equiv \{i_2, \dots, i_\tau\} \quad \text{and} \quad J \equiv \{j_2, \dots, j_\tau\}. \quad (1.97)$$

We use the familiar Hodges matrices in the ambitwistor representation

$$\Phi_{ij} = \langle \lambda_i \lambda_j \rangle S(z_i, z_j), \quad \Phi_{ii} = - \sum_{j \in p^- \setminus \{i\}} \Phi_{ij} \frac{\tilde{u}_j}{\tilde{u}_i} \quad (1.98)$$

and

$$\tilde{\Phi}_{ij} = [\tilde{\lambda}_i \tilde{\lambda}_j] S(z_i, z_j), \quad \tilde{\Phi}_{ii} = - \sum_{j \in p^+ \setminus \{i\}} \tilde{\Phi}_{ij} \frac{u_j}{u_i}, \quad (1.99)$$

for the integrand. The functions  $\lambda(z), \tilde{\lambda}(z)$  are defined as

$$\lambda(z) = \sum_{i \in p^-} \lambda_i \tilde{u}_i S(z, z_i), \quad \tilde{\lambda}(z) = \sum_{i \in p^+} \tilde{\lambda}_i u_i S(z, z_i), \quad (1.100)$$

where the locations  $z_i$  are determined by the scattering equations (1.18), while  $u_i, \tilde{u}_i$  are the corresponding scaling parameters.

It is well known that the ambi-twistor representation can be extended to  $\mathcal{N} \leq 3$  supersymmetry in a remarkably simple fashion. Given the Grassmann numbers  $\eta_i, \tilde{\eta}_i$  (transforming in the fundamental/anti-fundamental of the  $SU(\mathcal{N})$  R-Symmetry, respectively) from the external supermomenta, we can promote eq. (1.96) to a superamplitude

by including the factor

$$\exp \left( \sum_{\substack{i \in \mathbf{p}^- \\ i \in \mathbf{p}^+}} \eta_i \cdot \tilde{\eta}_j \tilde{u}_i u_j S(z_i, z_j) \right) , \quad (1.101)$$

whose behaviour under factorisation is simple and well understood [74]. This is astonishing not just because of its simplicity, but also because it makes space-time supersymmetry manifest. It is a consequence of the natural incorporation of onshell SUSY on twistor space.

### 1.4.1 Discussion

Having found the connected prescription formula for  $\mathcal{N} = 4$  sEYM, it is natural to ask whether we can construct a twistor-string based on [7] that computes it. It is worth pointing out that there is a natural way to incorporate the coupling of the gluons to gravitons, together with the Parke-Taylor factors, may be generated by inserting operators

$$\text{tr} \left( \bar{D}^{-1} \mathcal{O}_{\mathcal{A}} \bar{D}^{-1} \tilde{\mathcal{O}}_{\tilde{\mathcal{A}}} \right) .$$

Here  $\bar{D} = \bar{\partial} + \mathcal{A}(Z)$  and

$$\mathcal{O}_{\mathcal{A}} = \langle \rho, Z \rangle [\bar{\rho}, Y] \sim \langle \rho, Z \rangle \left[ \bar{\rho}, \frac{\partial \mathcal{A}}{\partial Z} \right], \quad \tilde{\mathcal{O}}_{\mathcal{A}} = \bar{\rho} \cdot Z \rho \cdot Y \sim \bar{\rho} \cdot Z \rho \cdot \frac{\partial \mathcal{A}}{\partial Z} ,$$

where  $\mathcal{A}$  and  $\tilde{\mathcal{A}}$  are gluon wavefunctions. This does however not account for the crucial ratio of Vandermonde factors, which is needed to render the interactions between gluons in different traces meaningful. In particular, a factorisation computation shows that its absence allows different traces to interact not only via gluons, but also via spurious vector modes, inherited from the  $\mathcal{N} = 8$  structure. We expect the correct model to require a modification of the worldsheet supersymmetry and the associated BRST ghost structure.

This concludes our exposition of  $\mathcal{N} = 4$  Einstein-Yang-Mills tree-level scattering amplitudes in the connected prescription. We now move on to the other side of the coin and describe a new set of ambitwistor string theories in the sense of [13].

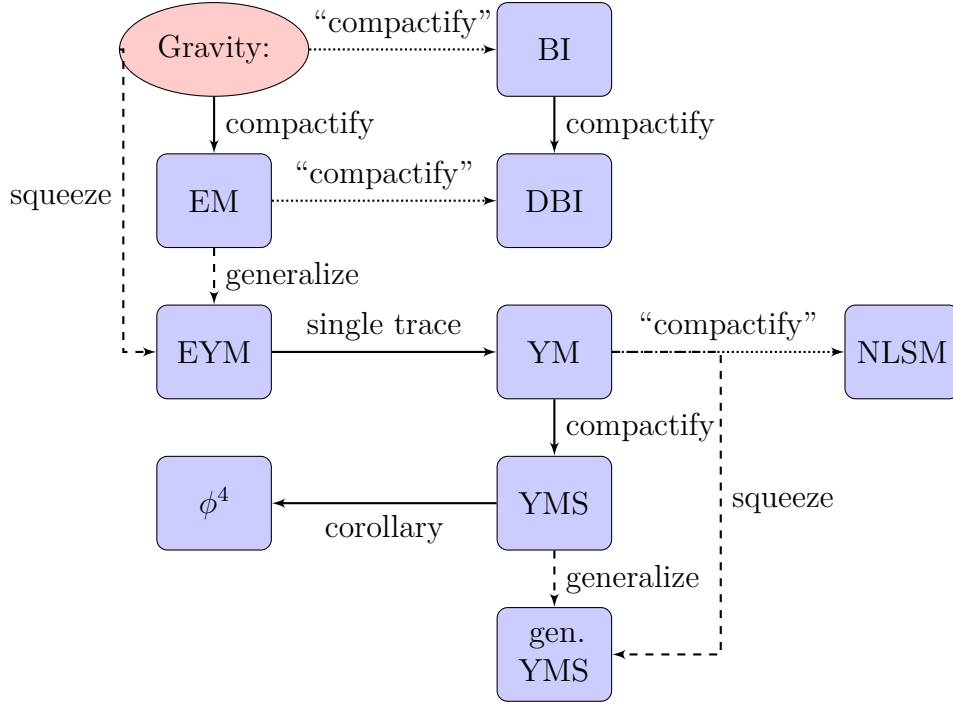


Fig. 1.1 Theories studied by CHY and operations relating them.

## 1.5 New Ambitwistor String Theories

In this section, we describe new ambitwistor string theories that give rise to the plethora of amplitude formulae introduced by Cachazo, He and Yuan. These include the original Einstein (E), Yang-Mills (YM) and biadjoint scalar (BS), together with new formulae for Einstein Maxwell (EM), Einstein-Yang-Mills (EYM), (Dirac)-Born-Infeld ((D)BI), Galileons (G), Yang-Mills Scalar (YMS) and nonlinear sigma model (NLSM) [72, 10], see figure 1.1 below for their diagram<sup>9</sup> of new amplitude formulae and relationships between them. They raised the challenge to find the underlying ambitwistor string theories that give rise to these formulae. Indeed, Ohmori in parallel work has already found the ambitwistor strings for the BI and Galileon theories [14]. Here we complement this list, explain some important details and highlight progress towards finding a consistent model for Einstein-Yang-Mills.

Ambitwistor strings are built out of a basic bosonic model together with worldsheet matter. The bosonic model leads to a framework in which the vertex operators required for amplitude calculations incorporate the scattering equations. The vertex operators also allow for the insertion of two currents  $v^l$  and  $v^r$  and these can be constructed from

<sup>9</sup>We thank CHY for permission to reproduce their diagram.

$S^l \backslash S^r$	$S_\Psi$	$S_{\Psi_1, \Psi_2}$	$S_{\rho, \Psi}^{(\tilde{m})}$	$S_{YM, \Psi}^{(\tilde{N})}$	$S_{YM}^{(\tilde{N})}$
$S_\Psi$	E				
$S_{\Psi_1, \Psi_2}$	BI	Galileon			
$S_{\rho, \Psi}^{(m)}$	EM $U(1)^m$	DBI	EMS $U(1)^m \times U(1)^{\tilde{m}}$		
$S_{YM, \Psi}^{(N)}$	EYM	ext. DBI	EYMS $SU(N) \times U(1)^{\tilde{m}}$	EYMS $SU(N) \times SU(\tilde{N})$	
$S_{YM}^{(N)}$	YM	Nonlinear $\sigma$	EYMS $SU(N) \times U(1)^{\tilde{m}}$	gen YMS $SU(N) \times SU(\tilde{N})$	Biadjoint Scalar $SU(N) \times SU(\tilde{N})$

Table 1.1 Theories arising from the different choices of matter models.

additional worldsheet matter (the natural choice for  $v^l$  and  $v^r$  in the bosonic model does not seem to lead to interesting amplitudes). The various Pfaffians, determinants or Parke-Taylor factors that are possible choices for the  $I^l$  and  $I^r$  arise as worldsheet correlators of currents for the  $v^l$  and  $v^r$  respectively. Corresponding to the five choices for the  $I^l$  and  $I^r$  in the CHY formulae we will introduce five choices of worldsheet matter, see table 1.1.

In the original models of [13] just two ingredients were used to construct  $I^l$  and  $I^r$ , worldsheet supersymmetry  $S_\Psi$ , and a current algebra  $S_J$ . Einstein, Yang-Mills and Biadjoint scalar theories were obtained from the choices  $(S^l, S^r) = (S_\Psi, S_\Psi)$ ,  $(S_\Psi, S_J)$  and  $(S_J, S_J)$  respectively. The current algebra  $S_J$  has the defect that it also leads to multi-trace terms in its correlators that were ignored by hand. Here we use a different worldsheet CFT, the comb system<sup>10</sup>,  $S_{CS}$ . This gives a new way to obtain colour factors together with their Parke-Taylor cyclic denominators in such a way that these multi-trace terms simply do not appear. Furthermore, the colour factors are presented not as cyclic single trace terms, but as strings of structure constants arranged in a ‘comb’, hence the name. However, the number of gauge particles in this system is doubled. To remedy this issue, a reduced system  $S_{YM}$  with the correct number of gauge particles can be constructed, but this is always anomalous. Nevertheless, it is sufficient to produce the correct tree amplitudes, and so we use this system instead

<sup>10</sup>This was originally introduced by David Skinner and Eduardo Casali [75] in the context of twistor-strings, but never published.



of the current algebra in the table 1.1. The remaining systems that we use will be combinations of these (with  $S_{\rho,\Psi}$  essentially being the abelian limit of the combination  $S_{CS,\Psi}$  of the comb system with worldsheet supersymmetry). These will be described in more detail in the remainder of this section.

There are a number of questions that one can ask about these models. For example, if they are critical and anomaly free, then one can attempt to calculate loop amplitudes by taking the correlation functions on higher genus Riemann surfaces as described in [26]. For this to work at 1-loop, we must check modularity. Another issue is as to whether there are any further vertex operators in the theories and if so, we can hope to extend the theory to include additional fields and calculate the corresponding amplitudes. This, in particular, happens in theories containing worldsheet supersymmetry  $S_\Psi$  and leads to supersymmetric extensions of the theories and amplitudes as described also in [26]. We will find a number of new critical theories and give a brief discussion of these issues in the conclusions section.

Potentially the most interesting of these models is that for Einstein-Yang-Mills. We obtain these in two forms. One gives the correct CHY tree-level amplitudes but is anomalous using  $S_{YM}$ . The other has vanishing central charge in 10 dimensions but has doubled gluons in the theory. The gauge theory part of the action is given by

$$S_{T^*YM} = \int d^D x \operatorname{tr}(a_\mu D_\nu F^{\mu\nu}), \quad (1.102)$$

and we refer to it as  $T^*YM$  as it describes a linearised Yang-Mills field  $a$  propagating on a full Yang-Mills background for the field  $A$  with curvature  $F$ . Here  $a$  is canonically conjugate to  $F$  hence the name  $T^*YM$  as opposed to  $TYM$ . This should give correct Yang-Mills amplitudes at one loop but has no higher loop amplitudes in the pure gauge sector. In its critical dimension  $d = 10$ , we would expect it to give a valid expression for the 1-loop integrand for Yang-Mills also.

Table 1.1, showing how the theories are determined in terms of a pair of worldsheet systems, is a remarkable manifestation of the notion of double copy. This notion has been explored mostly in the context of gravity amplitudes, which are obtained as the double copy of gauge theory ones [12, 76]. In the formalism of the scattering equations, this is the double copy of Pfaffian factors, and in ambitwistor string theory, this is the double copy of the worldsheet system  $S_\Psi$ , as in table 1.1. The amplitude formulae of ref. [10] and our results extend this notion to a range of other theories. Regarding the relation to previous work, we should mention that a double copy construction for Einstein-Yang-Mills amplitudes was first presented in [59] for the single trace contribution, and in [77] for the complete amplitude, with results also at loop level.

These double copy constructions are based on the colour-kinematics duality [76, 78], whose relation to the scattering equations has been explored in [57, 79, 80].

### 1.5.1 Worldsheet matter models and their correlators

In [13], two matter models were considered: (1)  $S_\rho$ , a current algebra which we will take to be generated by free fermions, and (2)  $S_\Psi$ , which introduces a degenerate worldsheet supersymmetry. This latter extends  $Q$  so as to change the choice of current  $v = \epsilon \cdot P$  in the bosonic model to one that we will want. These led to three models with  $(S^l, S^r)$  given by  $(S_{\Psi_l}, S_{\Psi_r})$  for type II supergravity,  $(S_\Psi, S_\rho)$  for Yang-Mills amplitudes and  $(S_{\rho_l}, S_{\rho_r})$  for amplitudes of a biadjoint scalar theory. In this work we will consider a third type of matter that we call the ‘comb system’  $S_{CS}$ , [75], a worldsheet conformal field theory that will be important for Yang-Mills amplitudes so called because its correlators give colour invariants in the form of comb structures built out of structure constants rather than colour traces. In the rest of this section, we describe these matter systems and the natural currents to which they give rise as candidates for  $v^l$  and  $v^r$  and their correlation functions. In the next section, we see how these are altered when these systems are combined.

#### Free fermions $S_\rho$ and current algebras $S_j$ .

The standard action for ‘real’ free fermions  $\rho^a \in K^{1/2}$ ,  $a = 1, \dots, m$ , is

$$S_\rho = \int \rho^a \bar{\partial} \rho^a,$$

(the summation convention is assumed). The term ‘real’ is used to distinguish them from the complex fermion system given by

$$S_{\rho, \tilde{\rho}} = \frac{1}{2\pi i} \int \tilde{\rho}_a \bar{\partial} \rho^a.$$

The simplest currents in the real case are  $j^{ab} = \rho^a \rho^b$  and form an elementary example of a current algebra for  $SO(m)$  (in the complex case  $j_b^a = \tilde{\rho}_b \rho^a$  generate a current algebra for  $SU(m)$ ).

More generally, we can consider an arbitrary current algebra  $j^a \in K \otimes \mathfrak{g}$ , where  $\mathfrak{g}$  is some Lie algebra,  $a = 1, \dots, \dim \mathfrak{g}$ , satisfying the usual current algebra OPE,

$$j^a(\sigma) j^b(0) \sim \frac{k \delta^{ab}}{\sigma^2} + \frac{i f^{abc} t^c(\sigma)}{\sigma} + \dots, \quad (1.103)$$

where  $f^{abc}$  are the structure coefficients,  $[t^a, t^b] = f^{abc}t^c$ ,  $\delta^{ab}$  is the Killing form, and  $k$  is the level. This could be constructed from free fermions, WZW models or some other construction and we will generally represent such matter as  $S_j$ .

Given choices of  $t \in \mathfrak{g}$ , the current algebra can contribute

$$v = t \cdot j$$

to one or both factors  $v_l$  and  $v_r$  of the vertex operators  $V$ . The current correlators  $\langle t_1 \cdot j_1 \dots t_n \cdot j_n \rangle$ , where  $j_i = j(\sigma_i)$ , lead to Park-Taylor factors:

$$PT(1, \dots, n) = \frac{\text{tr}(t_1 \dots t_n)}{\sigma_{12}\sigma_{23} \dots \sigma_{n1}},$$

where  $\sigma_{ij} = \sigma_i - \sigma_j$ . However, the correlators also lead to multi-trace terms that are ultimately problematic and unwanted.

### Worldsheet supersymmetry $S_\Psi$ .

Worldsheet supersymmetry is introduced by adding fermionic worldsheet spinor fields  $\Psi^\mu \in \mathbb{C}^d \otimes K^{1/2}$ , and a gauge field  $\chi \in \Omega^{(0,1)}(T^{1/2})$  for the supersymmetry. Their action is

$$S_\Psi = \frac{1}{2\pi i} \int \Psi \cdot \bar{\partial} \Psi + \chi P \cdot \Psi.$$

The constraint leads to worldsheet gauge transformations

$$\delta\chi = \bar{\partial}\eta, \quad \delta X = \eta\Psi, \quad \delta\Psi = \eta P, \quad \delta P = 0,$$

where  $\eta$  is a fermionic parameter. Gauge fixing leads to bosonic ghosts  $\gamma \in T^{1/2}$  and corresponding antighosts  $\beta$ . The BRST operator acquires an extra term

$$Q_\Psi = \oint \gamma \mathcal{G}_\Psi, \quad \mathcal{G}_\Psi := P \cdot \Psi.$$

On  $\mathbb{CP}^1$ , the ghosts  $\gamma$  have two zero modes. Thus, as far as the fermionic symmetry is concerned, we need two fixed vertex operators with one current factor of the form  $\delta(\gamma)$  multiplied by a field now with values in  $K^{1/2}$ , and then the ‘integrated’ ones (in the fermionic sense) arising from descent. The relevant currents are

$$u = \delta(\gamma)\epsilon \cdot \Psi, \quad v = \epsilon \cdot P + k \cdot \Psi \epsilon \cdot \Psi,$$

with just two of the  $u$ s required in a correlator.

These operators are invariant under the discrete symmetry that changes the sign of  $\Psi$ ,  $\chi$  and the ghosts. Imposing invariance under this symmetry will exclude mixing between the ingredients of these operators thought of as parts of  $S^l$  and others that might be part of  $S^r$ . We will refer to this as GSO symmetry.

The correlators of these currents lead to the reduced Pfaffians of CHY:

$$\langle u_1 u_2 v_3 \dots v_n \rangle = \text{Pf}'(M) = \frac{1}{\sigma_1 - \sigma_2} \text{Pf}(M_{12}),$$

where  $M$  is the skew  $2n \times 2n$  matrix with  $n \times n$  block decomposition

$$M = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \quad A_{ij} = \frac{k_i \cdot k_j}{\sigma_{ij}}, \quad B_{ij} = \frac{\epsilon_i \cdot \epsilon_j}{\sigma_{ij}},$$

and

$$C_{ij} = \frac{\epsilon \cdot k_j}{\sigma_{ij}}, \quad i \neq j, \quad C_{ii} = -\epsilon_i \cdot P(\sigma_i),$$

and  $M_{12}$  is  $M$  with the first two rows and columns removed.

### Comb system $S_{CS}$ .

The comb system [75] was introduced as a way of obtaining colour factors as sequences of contractions of structure constants rather than as colour ordered traces. In general, such contractions can be generated from trivalent diagrams with the structure constants  $f^{abc}$  of some Lie algebra at the vertices and contractions  $\delta^{ab}$  along the internal edges. It is well known that these are linearly dependent as a consequence of the Kleiss-Kuijff relations with a basis being given by ‘combs’, with  $n - 2$  vertices lined up in a row [81, 82] and end points given by 1 and  $n$ :

$$\begin{array}{c} 2 \quad 3 \quad \dots \\ | \quad | \quad | \quad \dots \quad | \quad | \quad | \\ 1 \text{ --- } | \text{ --- } | \text{ --- } | \text{ --- } | \text{ --- } | \text{ --- } | \text{ --- } n \end{array} \quad \rightarrow \quad f^{a_1 a_2 b_1} f^{b_1 a_3 b_2} \dots f^{b_{n-3} a_{n-1} a_n}.$$

The comb system has the remarkable property that, in conjunction with worldsheet supersymmetry, only these combs arise from correlators and not the multitrace terms that arise from an ordinary current algebra. This system arises from an action for matter fields  $\rho, \tilde{\rho}, q, y \in \mathfrak{g} \otimes K^{1/2}$  i.e., worldsheet spinors taking values in the Lie algebra

$\mathfrak{g}$  of some gauge group. The worldsheet action is

$$S_{CS} = \int \tilde{\rho} \cdot \bar{\partial} \rho + q \cdot \bar{\partial} y + \chi \rho \cdot \left( \frac{1}{2} [\rho, \tilde{\rho}] + [q, y] \right)$$

with  $\rho, \tilde{\rho}$  fermionic and  $q, y$  bosonic and the  $\cdot$  is used to denote the Killing form on the Lie algebra. As before,  $\chi$  is a gauge field on the worldsheet with values in  $T^{1/2} \otimes \Omega^{0,1}$  and we are gauging the current<sup>11</sup>  $\rho \cdot \left( \frac{1}{2} [\rho, \tilde{\rho}] + [q, y] \right)$ , which is a section of  $K^{3/2}$ . The gauging introduces transformations now for fermionic  $\alpha \in T^{1/2}$

$$\delta(\rho, \tilde{\rho}, q, y) = \alpha \left( \frac{1}{2} [\rho, \rho], [\rho, \tilde{\rho}], [\rho, q], [\rho, y] \right), \quad \delta \chi = \bar{\partial} \alpha.$$

As in the case of worldsheet supersymmetry, gauge fixing gives bosonic ghosts  $\gamma \in T^{1/2}$  and antighosts  $\beta$  with a contribution to the BRST operator of

$$Q_{CS} = \oint \gamma \mathcal{G}_{CS}, \quad \mathcal{G}_{CS} := \rho \cdot \left( \frac{1}{2} [\rho, \tilde{\rho}] + [q, y] \right).$$

As for  $S_\Psi$ , there are two zero-modes for the ghosts, and so we will need two fermionically fixed operators with the rest integrated. The currents that contribute to the vertex operators in this system now depend on a Lie algebra element  $t \in \mathfrak{g}$ , with two types of fixed and integrated ones respectively being

$$u = \delta(\gamma) t \cdot \rho, \quad \tilde{u} = \delta(\gamma) t \cdot \tilde{\rho}, \quad v = {}^{1/2} t \cdot [\rho, \rho], \quad \tilde{v} = t \cdot ([\rho, \tilde{\rho}] + [q, y]).$$

Here  $v = \{Q_{CS}, u\}$  and  $\tilde{v} = \{Q_{CS}, \tilde{u}\}$ , and, in any correlator, we need two fixed and the remaining unfixed vertex operators<sup>12</sup>. Notice that  $\tilde{v} = t \cdot j$ , where  $j^a$  is a level zero current algebra, and that

$$\tilde{v}(\sigma) t' \cdot \tilde{\rho}(0) \sim -\frac{[t, t'] \cdot \tilde{\rho}(0)}{\sigma} + \dots, \quad \tilde{v}(\sigma) t' \cdot \rho(0) \sim -\frac{[t, t'] \cdot \rho(0)}{\sigma} + \dots.$$

<sup>11</sup>With different assignment of worldsheet spins this current would be a normal BRST current. If we were to take  $\rho, y$ , scalars and  $\tilde{\rho}, q$  sections of  $K$ , then  $\rho$  and  $\tilde{\rho}$  could be taken to be the ghosts associated to gauge fixing a worldsheet gauge field  $a \in \Omega^{0,1} \otimes \mathfrak{g}$  with action  $\int_\Sigma q \cdot \bar{\partial} y + q \cdot [a, y]$ . This fact allows one to see the consistency of this current reasonably rapidly.

<sup>12</sup>A more symmetric way to understand this is to say that we choose all unintegrated vertex operators, but then we must insert  $n - 2$  ‘picture-changing operators’

$$\Upsilon = \delta(\beta) \rho \cdot \left( \frac{1}{2} [\rho, \tilde{\rho}] + [q, y] \right).$$

These could be inserted anywhere in general. If inserted at one of the  $u, \tilde{u}$  insertion points, it will convert it into a corresponding  $v, \tilde{v}$ . A similar approach can be taken for correlators associated with the  $S_\Psi$  matter system.

The correlators are as follows

**Proposition 1.5.1 (Casali-Skinner)** *Correlators of the currents  $u, v, \tilde{u}, \tilde{v}$  are only nonvanishing when there is just one untilded current and give*

$$\langle u_1 \tilde{v}_2 \dots \tilde{v}_{n-1} \tilde{u}_n \rangle = \langle \tilde{u}_1 v_2 \tilde{v}_3 \dots \tilde{v}_{n-1} \tilde{u}_n \rangle = \mathcal{C}(1, \dots, n)$$

where

$$\mathcal{C}_n = \mathcal{C}(1, \dots, n) := \frac{\text{tr}(t_1[t_2, [\dots, [t_{n-1}, t_n] \dots]])}{\sigma_{12}\sigma_{23} \dots \sigma_{n1}} + \text{Perm}(2, \dots, n-1).$$

Instead of giving the colour traces, we obtain ‘combs’, i.e., strings of structure constants  $\text{tr}(t_1[t_2, [\dots, [t_{n-1}, t_n] \dots]])$  as described in [83, 82].

The argument is as follows. The fact that we can have at most two  $u, \tilde{u}$ s is the standard counting of  $\gamma$  ghost zero modes. Consider the  $\rho, \tilde{\rho}$  contractions: that these are the only nontrivial correlators comes from the need to have as many  $\rho$ s as  $\tilde{\rho}$ , so it is easily seen that we can have only one untilded current which can either be a  $u$  or a  $v$ . The  $v, \tilde{v}$ s connect along a ‘comb’, whereas the  $u, \tilde{u}$ s form the ends. Such contractions connecting all  $n$  vertex operators form the right-hand side above. We can also have contractions in which a collection of  $\tilde{v}$ s come together in contractions to form a loop. This is where the  $(q, y)$  system comes into play. These can only form loops, but, being bosonic, their loop contractions cancel such loop contractions from the  $\rho, \tilde{\rho}$  system. This can also be seen from the form of the current algebra generated by the  $\tilde{v}$ s. This has by construction level zero so that, after a sequence of OPE’s, cannot generate a nontrivial trace.

### Other systems with comb structure, $S_{YM}$ .

A problem with the CS system above is that there are clearly two types of gluons, tilded and untilded corresponding to the vertex operators  $(\tilde{u}, \tilde{v})$  and  $(u, v)$  respectively. We will see that this is not appropriate for pure Yang-Mills although it does give a theory that is sufficient to generate Einstein-YM tree amplitudes correctly on certain trace sectors, the ones selected by the choice of untilded operators.<sup>13</sup> The system we introduce here will give the complete Einstein Yang-Mills amplitude from a single correlator but will be anomalous.

<sup>13</sup>One may try to symmetrise the correlator in tilded versus untilded gluonic operators, for instance by using  $u_t + \tilde{u}_t$  and  $v_t + \tilde{v}_t$ , but then there will be an over-counting of contributions, so that the relative factors of different terms are not correct.

A worldsheet CFT that will generate YM following the ideas above requires the following ingredients. We need a fermionic worldsheet spinor  $\rho^a \in \mathfrak{g}$  for the fixed vertex operator, a current algebra  $v^a \in \mathfrak{g}$  at level zero for the integrated one; the level zero allows us to avoid multitrace terms and loops. Finally we need a spin 3/2 current  $\mathcal{G}_{YM}$  with the following OPE to give the appropriate group compatibilities and descent:

$$\rho^a(\sigma)\rho^b(0) \sim \frac{\delta^{ab}}{\sigma}, \quad v^a(\sigma)\rho^b(0) \sim \frac{f^{abc}\rho^c(0)}{\sigma}, \quad \mathcal{G}(\sigma)\rho^a(0) \sim \frac{v^a(0)}{\sigma}, \quad \mathcal{G}(\sigma)\mathcal{G}(0) \sim 0. \quad (1.104)$$

It is easy to see that this can be partially realised with  $\rho^a$  a ‘real’ free fermion with action  $\frac{1}{2} \int \rho^a \bar{\partial} \rho^a$  and with

$$v^a = -\frac{1}{2} f^{abc} \rho^b \rho^c + j^a, \quad j^a(\sigma)\rho^b(0) \sim 0,$$

we will obtain the first two of the equations above. In order for  $v^a$  to be a current algebra with level zero, because  $\frac{1}{2} f^{abc} \rho^b \rho^c$  is a current algebra with level  $-C$  where  $f^{abc} f^{\bar{a}bc} = C \delta^{a\bar{a}}$ , we must take  $j^a$  to be a current algebra with level  $k = C$ . There are many ways to do this, so let us leave this to one side for a moment. We then need to construct  $\mathcal{G}$ . In order for  $\mathcal{G}$  to generate  $v^a$  from  $\rho^a$ , we must have

$$\mathcal{G} = -\frac{1}{6} f^{abc} \rho^a \rho^b \rho^c + \rho^a j^a + \dots$$

where the  $\dots$  has nonsingular OPE with  $\rho^a$  and  $j^a$ . At this point, however, we see that an anomaly arises preventing  $\{\mathcal{G}, \mathcal{G}\} = 0$ . To be specific,

$$\mathcal{G}(\sigma)\mathcal{G}(0) \sim \frac{C \dim(G)}{\sigma^3} + \frac{j^a j^a(0)}{\sigma},$$

where we recall that the energy-momentum tensor of the current algebra  $j$  is given by  $T(\sigma) = :j^a j^a(\sigma): / 2k$ . Therefore, we are able to satisfy the first three equations of (1.104), while the last equation is anomalous.

### Central charges

We remark that the theories  $S_B$ ,  $S_\rho$ ,  $S_\Psi$  and  $S_{CS}$  above respectively have central charges

$$c_B = 2d - 52, \quad c_\rho = m/2, \quad c_\Psi = d/2 + 11, \quad c_{CS} = 11,$$

the latter being just that of the  $\beta - \gamma$  system as the  $\dim G$  parts cancel via supersymmetry. (This can be different if the  $(q, y)$  are not taken to be spin  $1/2$ .) Notably, the type II supergravity model is critical in 10 dimensions as then  $c_B + 2c_\Psi = 0$ . These considerations are less interesting for  $S_{YM}$  as that theory is already quite anomalous, and in any case, its central charge will depend on the choice of current algebra  $j^a$ .

### 1.5.2 Combined Matter models

On their own, the new worldsheet matter theories  $S_{CS}$  and  $S_{YM}$  of the previous section do little more than giving an alternative to the current algebras in the original models of [13] that avoids the multitrace terms that were neglected by hand. To obtain new theories, we will consider the contributions to  $S^l$  or  $S^r$  of combinations of the above matter systems. Even without  $S_{CS}$  and  $S_{YM}$ , we will obtain a number of interesting new models. Here we will consider the allowable vertex operators and the correlators of the various combinations that we can form. These are summarised in the table 1.2.

	Fermionic current $\mathcal{G}$	Matter	Vertex operators	Correlator
$S_\Psi$	$P \cdot \Psi$	$\Psi$	$u_\Psi = \delta(\gamma) \epsilon \cdot \Psi$	$\text{Pf}'(M)$
$S_{\Psi_1, \Psi_2}$	$P \cdot \Psi_1$ $P \cdot \Psi_2$	$\Psi_1, \Psi_2$	$u_{\Psi_1} = \delta(\gamma_2) k \cdot \Psi_1$ $u_{\Psi_2} = \delta(\gamma_1) k \cdot \Psi_2$	$(\text{Pf}'(A))^2$
$S_{\rho, \Psi}$	$P \cdot \Psi$	$\Psi, \rho_a \quad a = 1, \dots, m$	$u_\Psi = \delta(\gamma) \epsilon \cdot \Psi$ $u_{\rho_a} = \delta(\gamma) \rho_a$	$\text{Pf}(\chi) \text{Pf}'(M _{\text{red}})$
$S_{CS, \Psi}$	$P \cdot \Psi + \text{tr}(\rho(\frac{1}{2}[\tilde{\rho}, \rho] + [q, y]))$	$\Psi, (\tilde{\rho}, \rho), (q, y)$	$u_\Psi = \delta(\gamma) \epsilon \cdot \Psi$ $\tilde{u}_{CS} = \delta(\gamma) \text{tr}(t\tilde{\rho})$ $u_{CS} = \delta(\gamma) \text{tr}(t\rho)$	$\mathcal{C}_{(1)} \dots \mathcal{C}_{(m)} \text{Pf}'(\Pi)$
$S_{CS}$	$\text{tr}(\rho(\frac{1}{2}[\tilde{\rho}, \rho] + [q, y]))$	$(\tilde{\rho}, \rho), (q, y)$	$\tilde{u}_{CS} = \delta(\gamma) \text{tr}(t\tilde{\rho})$ $u_{CS} = \delta(\gamma) \text{tr}(t\rho)$	$\mathcal{C}_n$

Table 1.2 Table of matter models, their combinations and worldsheet correlators



$S_{\rho,\Psi}$

Here we take

$$S^l = S_{\rho,\Psi} := S_\rho + S_\Psi.$$

Although the free fermion system  $S_\rho$  would seem to naturally lead to the  $SO(m)$  current algebra  $j^{ab} = j^{[ab]} = \rho^a \rho^b$ , and therefore superficially be thought to give the same results as the current algebra, in the presence of worldsheet supersymmetry, the currents  $j^{ab}$  as constituents of vertex operators are not BRST invariant, since

$$\{Q_\Psi, j^{ab} e^{ik \cdot X}\} = ik \cdot \Psi j^{ab} e^{ik \cdot X} \neq 0.$$

However, in this context allowable fixed and integrated currents are respectively

$$u^a = \delta(\gamma) \rho^a, \quad v^a = k \cdot \Psi \rho^a, \quad a = 1, \dots, m.$$

We also have the standard BRST invariant currents from  $S_\Psi$ , which in this context we will denote  $u_\epsilon = \delta(\gamma) \epsilon \cdot \Psi$  and  $v_\epsilon = \epsilon \cdot P + k \cdot \Psi \epsilon \cdot \Psi$ .

In general we will be concerned with a correlator  $\langle u_1 u_2 v_3 \dots v_n \rangle$  where, if  $(\gamma, h)$  is a partition of  $1, \dots, n$ , for  $i \in \gamma$  the current will be one of the new photon currents, and for  $i \in h$  it will be a  $S_\Psi$  current depending on a polarisation vector  $\epsilon_\mu$ . The correlator will factorise into one for the constituent  $\rho$ 's and one for the  $\Psi$ 's. We compute these as Pfaffians of the associated matrices of possible contractions in the correlator. The simplest is the  $\rho$  system. If we restrict it to take values in an algebra with vanishing structure constants, e.g.  $\oplus^m \mathfrak{u}(1)$ , the OPEs lead to the  $|\gamma| \times |\gamma|$  CHY matrix

$$\mathcal{X}_{ij} = \frac{\delta^{a_i a_j}}{\sigma_{ij}}, \quad i, j \in \gamma, \quad i \neq j, \quad \text{otherwise} \quad \mathcal{X}_{ij} = 0.$$

The Kronecker delta  $\text{tr}(t^{a_i} t^{a_j})$  in the numerator ensures only photons of the same flavour interact.

Much as before, the  $\Psi$  system leads to the matrix of possible  $\Psi$  contractions

$$M_{\text{Red}} = \begin{pmatrix} A^{\gamma\gamma} & A^{\gamma h} & -C^{h\gamma T} \\ A^{h\gamma} & A^{hh} & -C^{hhT} \\ C^{h\gamma} & C^{hh} & B \end{pmatrix},$$

where we have divided the matrix into the block decomposition under  $n = |\gamma| + |h|$  and

$$A_{ij} = \frac{k_i \cdot k_j}{\sigma_{ij}}, i \neq j, \quad A_{ii} = 0, \quad B_{ij} = \frac{\epsilon_i \cdot \epsilon_j}{\sigma_{ij}}, \quad i, j \in h, i \neq j.$$

and

$$C_{ij} = \frac{\epsilon_i \cdot k_j}{\sigma_{ij}}, \quad i \in h, i \neq j.$$

Finally, the additional  $\epsilon \cdot P$  term in the  $S_\Psi$  vertex operator is incorporated by setting  $C_{ii} = -\epsilon_i \cdot P(\sigma_i)$  as before. In this case, we obtain a reduced Pfaffian associated with the two fixed vertex operators as before. Our final correlator expression is therefore

$$\langle u_1 u_2 v_3 \dots v_n \rangle = \text{Pf}(\mathcal{X}) \text{Pf}'(M_{Red}).$$

Now for the GSO symmetry, we require all fields,  $\rho, \Psi$  and the ghosts to change sign simultaneously.

$S_{\Psi_1, \Psi_2}$

Here we take two worldsheet supersymmetries

$$S^l = S_{\Psi_1} + S_{\Psi_2}.$$

There are two contributions to the BRST operator,  $Q_{\Psi_1} + Q_{\Psi_2}$ . The normal currents from  $S_{\Psi_1}$  and  $S_{\Psi_2}$  are no longer invariant as, for example,

$$Q_{\Psi_2} \delta(\gamma_1) \epsilon \cdot \Psi_1 e^{ik \cdot X} = \delta(\gamma_1) k \cdot \Psi_2 \epsilon \cdot \Psi_1 e^{ik \cdot X} \neq 0.$$

However, the nontrivial BRST invariant currents are descendants simply of  $\delta(\gamma_1)\delta(\gamma_2)$  so

$$u = \delta(\gamma_1)\delta(\gamma_2), \quad v = k \cdot \Psi_1 k \cdot \Psi_2,$$

as given in [14] (and we also have partial descendants  $\delta(\gamma_1)k \cdot \Psi_2$  and  $\delta(\gamma_2)k \cdot \Psi_1$ ). Again, the correlator of  $n$  such vertex operators factorises into a product of the Pfaffians of the matrix of all possible  $\Psi_1$  contractions and that for all  $\Psi_2$  contractions. These matrices are given simply by the  $A$  matrix with off-diagonal entries  $k_i \cdot k_j / \sigma_{ij}$  as before. This matrix has co-rank two, and we take a reduced Pfaffian (corresponding to the choice of fixed versus integrated vertex operators). We therefore now obtain

$$\langle u_1 u_2 v_3 v_4 \dots v_n \rangle = \text{Pf}'(A)^2.$$

One might ask whether one can carry on to combine three or more  $S_\Psi$  systems, but this is not possible on one side, that is, to produce nontrivial BRST invariant currents.

Again for the GSO symmetry, we require all fields,  $\Psi_1, \Psi_2$  and the ghosts to change sign simultaneously.

### $S_{YM,\Psi}$

In Sections 1.5.1 and 1.5.1, we introduced  $S_{CS}$  and  $S_{YM}$  whose correlators provide the colour comb-structure together with Parke-Taylor factors. For the remainder of this section, we will combine each of these two systems with  $S_\Psi$ . The goal is to obtain the building block of Einstein-Yang-Mills amplitudes that gives the appropriate interactions between gluons and gravitons. We start by discussing the combined theory  $S_{YM,\Psi}$ , which is slightly simpler than  $S_{CS,\Psi}$  and possesses the main important features. Despite  $S_{YM}$  not being quantum-mechanically consistent - and this problem extends to  $S_{YM,\Psi}$  - we are able to obtain tree amplitudes. The theory  $S_{CS,\Psi}$  can be made consistent, but has two types of gluons and the corresponding amplitudes arise from an action that is not Yang-Mills (although it contains its classical solutions).

Since both worldsheet matter theories  $S_\Psi$  and  $S_{YM}$  involve the gauging of spin 3/2 currents  $\mathcal{G}_\Psi = P \cdot \Psi$  and  $\mathcal{G}_{YM} = \rho \cdot \left(-\frac{1}{6}[\rho, \rho] + j\right)$ , we have the option of gauging both these currents together or separately. If we gauge them separately, we find that the resulting system is too restrictive to lead to interesting results. Thus we gauge the sum

$$\mathcal{G} = P \cdot \Psi + \rho \cdot \left(-\frac{1}{6}[\rho, \rho] + j\right),$$

perform gauge fixing and introduce a single set of ghosts  $(\beta, \gamma)$ . We find that the currents

$$u_t = \delta(\gamma)\rho \cdot t, \quad u_\epsilon = \delta(\gamma)\epsilon \cdot \Psi$$

still give us allowed fixed vertex operators. BRST descent leads to the integrated vertex operators

$$v_t = k \cdot \Psi \rho \cdot t + v_t^0, \quad v_\epsilon = \epsilon \cdot P + \epsilon \cdot \Psi k \cdot \Psi,$$

where  $v_t^0$  denotes the original  $S_{YM}$  integrated vertex operator, satisfying the OPE relations (1.104) except the last. Although the failure of the last relation means that the BRST quantisation is inconsistent, the correlator of the vertex operators does nevertheless give the correct amplitudes.

In the previous section, we saw that the system  $S_{YM}$  on its own gives the correct colour-dressed Parke-Taylor factors, in terms of a comb structure. The combination with  $S_\Psi$  leads to additional insertions of  $\rho \cdot t$  and these will start additional combs. In this way, we obtain multiple colour combs/traces and get the right interactions with gravity states. On the other hand, the system  $S_\Psi$  on its own leads to a reduced Pfaffian. The combination with  $S_{YM}$  will lead to a different but closely related Pfaffian that incorporates the multi-comb structure. We now describe the complete correlator.

**Theorem 1** *As in [10], let the sets  $g$  index the gluons with vertex operators  $u_t, v_t$ , and  $h$  the gravitons with vertex operators  $u_\epsilon, v_\epsilon$ . To be non-zero, a correlator must contain two fixed vertex operators  $u$ 's, with the remaining ones being  $v$ 's. The correlator is then a sum over all partitions of the gluons into sets  $T_1, T_2, \dots, T_m$ , where  $\cup_{i=1}^m T_i = g$  and  $|T_i| \geq 2$ . Each partition gives rise to the term*

$$\sum_{\substack{c_1 < d_1 \in T_1 \\ \vdots \\ c_m < d_m \in T_m}} \mathcal{K}(c_1, d_1 | T_1) \cdots \mathcal{K}(c_m, d_m | T_m) \text{ Pf}' \left( \begin{array}{ccc|c} A_{ab} & A_{ac_j} & A_{ad_j} & (-C^T)_{ab} \\ A_{c_i b} & A_{c_i c_j} & A_{c_i d_j} & (-C^T)_{c_i b} \\ A_{d_i b} & A_{d_i c_j} & A_{d_i d_j} & (-C^T)_{d_i b} \\ \hline C_{ab} & C_{ac_j} & C_{ad_j} & B_{ab} \end{array} \right). \quad (1.105)$$

Here,  $a, b$  label gravitons and  $c_i, d_i$  label gluons in  $T_i$ , so that  $A_{ab}$  is an  $|h| \times |h|$  matrix,  $A_{c_i b}$  is an  $m \times |h|$  matrix, and  $A_{c_i c_j}$  is an  $m \times m$  matrix. Moreover, we defined

$$\mathcal{K}(i, j | T) = \sigma_{ji} \mathcal{C}(T), \quad (1.106)$$

where  $\mathcal{C}(T)$  is  $\mathcal{C}_n$  restricted to  $g \in T$ . The reduced Pfaffian  $\text{Pf}'$  is defined in eq. (A.15).

The proof is given in chapter A. This correlator reproduces the main building block of the CHY formula for Einstein-Yang-Mills amplitudes in [10]. Although not quite in the same form, the equivalence can easily be seen from Eqs. (3.16) and (3.17) of [10] and this form is more natural from its derivation as a correlator.

$S_{CS, \Psi}$

While  $S_{YM, \Psi}$  gives the correct amplitude, its BRST quantisation is inconsistent. We can obtain the same structure from  $S_{CS, \Psi}$  by combining the worldsheet theories  $S_\Psi$  and

$S_{CS}$ , which has the advantage of being anomaly free but the disadvantage of containing two types of gluons.

As for  $S_{YM,\Psi}$  we gauge the sum of spin 3/2 currents  $\mathcal{G}_\Psi = P \cdot \Psi$  and  $\mathcal{G}_{CS} = \rho \cdot \left(\frac{1}{2}[\rho, \tilde{\rho}] + [q, y]\right)$ , introducing the action

$$S_{CS\Psi} = \int \Psi \cdot \bar{\partial}\Psi + \tilde{\rho} \cdot \bar{\partial}\rho + q \cdot \bar{\partial}y + \chi \left( P \cdot \Psi + \rho \cdot \left(\frac{1}{2}[\rho, \tilde{\rho}] + [q, y]\right) \right).$$

Now the Lie-algebra valued fermion  $\rho$  is complex (i.e., not equal to  $\tilde{\rho}$ ), unlike the previous case of  $S_{YM,\Psi}$ . This will change the physical content of the model. The gauge fixing of  $\chi$  introduces just one set of ghosts  $(\beta, \gamma)$ , and we find the standard fixed currents for  $S_{CS}$  and  $S_\Psi$ ,

$$u_t = \delta(\gamma)\rho \cdot t, \quad \tilde{u}_t = \delta(\gamma)\tilde{\rho} \cdot t, \quad u_\epsilon = \delta(\gamma)\epsilon \cdot \Psi.$$

The BRST descent leads to the following currents

$$v_t = k \cdot \Psi \rho \cdot t + v_t^0, \quad \tilde{v}_t = k \cdot \Psi \tilde{\rho} \cdot t + \tilde{v}_t^0, \quad V_\epsilon = \epsilon \cdot P + \epsilon \cdot \Psi k \cdot \Psi,$$

where  $v_t^0$  and  $\tilde{v}_t^0$  denote the original  $S_{CS}$  integrated vertex operators, so that  $v_t$  and  $\tilde{v}_t$  acquire a new term in  $\Psi$ .

To impose GSO symmetry, we require invariance under flipping the sign of the fields  $\rho, \tilde{\rho}, q, y, \Psi, \chi$  and the corresponding ghosts.

Since we have untilded vertex operators  $u_t, v_t$ , and tilded ones  $\tilde{u}_t, \tilde{v}_t$ , the correlator will depend not only on the number of gluonic vertex operators versus gravity ones  $u_\epsilon, v_\epsilon$ , but also on the choice of whether the gluonic operators are of un-tilded or tilded type. Recall from the previous section that, for the theory  $S_{CS}$  on its own, the only non-vanishing correlators were those with a single untilded operator and this led to a single comb colour structure that is equivalent to a single trace term. This followed because of the need to have the same number of  $\rho$ s and  $\tilde{\rho}$ s in a nontrivial correlator and a single  $\tilde{\rho}$  could only arise in one or both of the two fixed vertex operator. Now single  $\tilde{\rho}$ s appear in  $\tilde{v}_t$  and this essentially represents the coupling to gravity. Thus the coupling to gravity introduces multiple trace terms, with the interaction between each single trace structure being mediated by gravity. It is easy to see that with the  $S_{CS,\Psi}$  system we can now have as many untilded vertex operators as we like with their number corresponding precisely to the number of traces.

**Theorem 2** *Let the set  $g$  index the gluons and  $h$  the gravitons. To be non-vanishing, a  $S_{CS,\Psi}$  correlator must have two fixed vertex operators, with the remaining ones*

integrated. The correlator of such a collection of vertex operators is a sum over all partitions of the gluons into sets  $T_1, T_2, \dots, T_m$ , where  $m$  is the number of untilded gluonic vertex operators, and such that there is only one such vertex operator per  $T_i$ ,  $\cup_{i=1}^m T_i = g$ ,  $|T_i| \geq 2$ . Each allowed partition gives a contribution equal to (1.105).

Thus the correlator is the same as for  $S_{YM,\Psi}$ , except that there is a restriction on the allowed partitions of the gluons into traces.

### 1.5.3 New Ambitwistor String Theories

We can now assemble the full table of theories by combining the various possible choices of matter models on the left and right. These can be identified with their corresponding space-time theories by comparing the correlators to the formulae of CHY, and this results in table 1.3. Hopefully the acronyms for the models are self-explanatory except perhaps that BS denotes the bi-adjoint scalar  $\phi^{aa'}$ , where  $a$  and  $a'$  are respectively indices for the Lie algebras of  $SU(N)$  and  $SU(N')$ , with action

$$S_{BS} = \int d^D x \left( -\frac{1}{2} \partial_\mu \phi^{aa'} \partial^\mu \phi^{aa'} + \frac{1}{6} \phi^{aa'} \phi^{bb'} \phi^{cc'} f^{abc} f^{a'b'c'} \right),$$

where  $f^{abc}$  and  $f^{a'b'c'}$  are the structure constants of  $SU(N)$  and  $SU(N')$  respectively.

$\begin{array}{c} S^r \\ \hline S^l \end{array}$	$S_\Psi$	$S_{\Psi_1, \Psi_2}$	$S_{\rho, \Psi}^{(m')}$	$S_{YM, \Psi}^{(N')}$	$S_{YM}^{(N')}$
$S_\Psi$	E				
$S_{\Psi_1, \Psi_2}$	BI	Galileon			
$S_{\rho, \Psi}^{(m)}$	EM $ _{U(1)^m}$	DBI	EMS $ _{U(1)^m \otimes U(1)^m}$		
$S_{YM, \Psi}^{(N)}$	EYM	extended DBI	EYMS $ _{SU(N) \otimes U(1)^{m'}}$	EYMS $ _{SU(N) \otimes SU(N')}$	
$S_{YM}^{(N)}$	YM	NLSM	YMS $ _{SU(N) \otimes U(1)^{m'}}$	gen. YMS $ _{SU(N) \otimes SU(N')}$	BS

Table 1.3 Theories arising from the different choices of matter models.

Galileon theories are described by the action

$$S_{\text{Galileon}} = \int d^D x \left( -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \sum_{m=3}^{\infty} g_m \phi \det \{ \partial^{\mu_i} \partial_{\nu_j} \phi \}_{i,j=1}^{m-1} \right),$$

where  $g_m$  are freely prescribable parameters. However, our amplitudes only have one parameter. The theory that is singled out by our model is the one described in [84] in four dimensions, and in [10, 85] in general dimension, which has smoother soft behaviour than the generic ones.

The Born-Infeld action is

$$S_{\text{BI}} = \int d^D x \frac{1}{\ell^2} \left( \sqrt{-\det(\eta_{\mu\nu} - \ell^2 F_{\mu\nu})} - 1 \right),$$

the Dirac Born-Infeld is

$$S_{\text{DBI}} = \int d^D x \frac{1}{\ell^2} \left( \sqrt{-\det(\eta_{\mu\nu} - \ell^2 \partial_\mu \phi^a \partial_\nu \phi^a - \ell F_{\mu\nu})} - 1 \right),$$

and the nonlinear-sigma model is

$$S_{\text{NLSM}} = \int d^D x \left( -\frac{1}{2} \text{tr} \left( (\mathbb{1} - \lambda^2 \Phi)^{-1} \partial_\mu \Phi (\mathbb{1} - \lambda^2 \Phi)^{-1} \partial^\mu \Phi \right) \right),$$

where  $\Phi = \phi^a t^a$ .

In the table 1.3, we have only used  $S_{YM}$ . Although this is sufficient to produce the correct tree-level amplitudes, it is an anomalous matter system and so has no hope to be extended beyond tree-level, and indeed its meaning as a string theory is unclear even at tree level. We can obtain the same tree-amplitudes up to combinatorial factors by use of the comb system  $S_{CS}$  and this is not anomalous. However, this does lead to a doubling of the gauge degrees of freedom as described below in detail for the Einstein Yang-Mills system and bi-adjoint scalar.

In table 1.4 we list the vertex operators in each model and the central charges. Setting the central charge to zero gives the models that are critical and for which there is some reasonable hope that loop integrands can be described via these theories if they prove to be modular.

### Einstein Yang-Mills and $T^*$ YM

The worldsheet model that we discussed in the context of Einstein-Yang-Mills theory,  $S_{CS,\Psi}$ , has a consistent quantisation. On the other hand, it does not correspond strictly to the building block of Einstein-Yang-Mills amplitudes, because only trace/comb structures consistent with the choice of un-tilde vertex operators are allowed. Attempts to find a theory that reproduces this correlator seem to lead back to the anomalous  $S_{YM,\Psi}$  system.

Since the theory  $S_{CS,\Psi}$  presents no problems and has correlators which match part of the Einstein-Yang-Mills building block, it is natural to ask whether it is related to a known theory. This theory must contain two types of gluons, associated to tilded and un-tilded vertex operators, and the un-tilded type must give the number of allowed multiple trace terms in an amplitude. These conditions are satisfied by the following spacetime action for the gauge field

$$S_{T^*\text{YM}} = \int d^D x \text{tr}(a_\mu D_\nu F^{\mu\nu}). \quad (1.107)$$

The field  $a_\mu$  is a Lagrange multiplier enforcing the Yang-Mills equations,  $D_\nu F^{\mu\nu} = 0$ , and the action can be seen as a linearization of the Yang-Mills action,  $A_\mu \rightarrow A_\mu + a_\mu$ . The field  $A_\mu$  corresponds to the tilded degrees of freedom, and the field  $a_\mu$  corresponds to the un-tilded ones. Since the propagator of this action connects  $a_\mu$  to  $A_\mu$  and the vertices contain a single  $a_\mu$ , the Feynman rules and a straightforward graph-theoretic argument show that there is one and only one  $a_\mu$  external field per trace, also when the system is minimally coupled to gravity.

### Bi-adjoint scalar

The use of the worldsheet system  $S_{CS}$ , with its two types of coloured currents,  $\tilde{v}$  and  $v$ , is the reason for the Lagrange-multiplier-type action (1.107). An even simpler example is the bi-adjoint scalar theory, BS in table 1.3. In this case, we can easily apply the procedure of [47] and obtain explicitly the equations of motion. As in that construction, which was concerned with the Einstein theory, the spacetime background fields modify the worldsheet theory only through the constraints. The deformation of the constraints in the bi-adjoint scalar theory is particularly simple: the deformed ambitwistor constraint becomes

$$\mathcal{H} = P^2 \quad \rightarrow \quad \mathcal{H}_{(\phi,\Phi)} = P^2 + \Phi^{aa'} \tilde{v}^a \tilde{v}'^{a'} + \phi^{aa'} v^a v'^{a'}, \quad (1.108)$$

where we introduced currents for each of the two independent groups  $SU(N)$  and  $SU(N')$ . The equations of motion are obtained as anomalies obstructing the vanishing of the constraint at the quantum level,

$$\begin{aligned} \mathcal{H}_{(\phi,\Phi)}(\sigma) \mathcal{H}_{(\phi,\Phi)}(0) &\sim \frac{1}{\sigma^2} \left( (2 \partial^\mu \partial_\mu \Phi^{aa'} + f^{abc} f^{a'b'c'} \Phi^{bb'} \Phi^{cc'}) \tilde{v}^a \tilde{v}'^{a'} \right. \\ &\quad \left. + (2 \partial^\mu \partial_\mu \phi^{aa'} + 2 f^{abc} f^{a'b'c'} \Phi^{bb'} \phi^{cc'}) v^a v'^{a'} \right) (0) \\ &\quad + \text{simple pole.} \end{aligned} \quad (1.109)$$



If the equations of motion hold, there is no double pole and in fact the OPE is finite, because there can be no simple pole in the self-OPE of a bosonic operator in the absence of higher poles. The spacetime action associated to these equations of motion takes the Lagrange-multiplier form

$$S_{\text{BS}} = \int d^D x \, \phi^{aa'} \left( \partial^\mu \partial_\mu \Phi^{aa'} + \frac{1}{2} f^{abc} f^{a'b'c'} \Phi^{bb'} \Phi^{cc'} \right). \quad (1.110)$$

It should be seen as the analogue of the gauge theory action (1.107).

#### 1.5.4 Discussion

There are many issues to explore further and we briefly mention a few of them here. We have listed the central charges for the various models that are not already anomalous in table 1.4. It can be seen that many of the models have some critical dimension where the central charge vanishes. Indeed, one can often simply add some number of Maxwell fields to make them critical if one starts in low enough dimension. This suggests that a number of these models might give rise to plausible string expressions for corresponding loop integrands such as given in [26] for the type II theory in 10 dimensions. However, an independent criterion is that the loop integrand so obtained should be modular invariant and this may well exclude many of the critical models as it does in conventional string theory.

There is also the question as to whether there are further vertex operators that we have missed and therefore further sectors of these theories. For the ten-dimensional models, following [26], one can introduce a spin field  $\Theta^\alpha$  associated to each  $\Psi$  field and use these to introduce further vertex operators that will correspond to space-time fields with spinor indices. For the type II Einstein theory, these give rise to the Ramond sector vertex operators [26] and it can be seen that the same procedure can be applied more generally to some of the models here, particularly the Einstein  $T^*$ YM models. Following the same procedure one then extends the Einstein NS sector to include the Ramond sectors of type II gravity theories. However, we can see that the  $T^*$ YM vertex operators can only be extended in this way on the one side corresponding to the spin operator constructed from the  $\Psi$  in the Yang-Mills vertex operator. Thus one supersymmetry acts trivially on the Yang-Mills and hence is degenerate (it does not square to provide the Hamiltonian on the Yang-Mills fields).

By extending the worldsheet matter fields, we have generated new possible couplings to space-time fields. It would be interesting to explore whether these couplings can be made consistent in the fully nonlinear regime as described in [47, 86].

There remain other formulae based on the scattering equations, for which an underlying ambitwistor string theory has not yet been found. It would for example be interesting to find ambitwistor strings that give rise to the class of formulae with massive legs [87–90], and that for ABJM theory [91, 92], although see the twistor string [93].

Perhaps the most irritating issue is that we have not been able to find an Einstein-Yang-Mills model that is anomaly-free without unwanted linearised modes. Conventional string theory produces such amplitudes in open string theory and in closed string heterotic models. However, the ambitwistor heterotic string has corrupt gravity amplitudes, and so far there has been no ambitwistor analogue of open strings. Nevertheless, the  $T^*$ YM model is likely to make sense and provide the correct amplitudes at 1-loop if modular, although the pure gauge sector does not have loop amplitudes beyond 1-loop.

Theories	Integrated vertex operators	Central charge $c$
E	$V_h = (\epsilon \cdot P + k \cdot \Psi \epsilon \cdot \Psi) \left( \tilde{\epsilon} \cdot P + k \cdot \tilde{\Psi} \tilde{\epsilon} \cdot \tilde{\Psi} \right)$	$3(d - 10)$
EM	$V_h, V_\gamma$ $V_\gamma = (k \cdot \Psi t \cdot \rho) \left( \tilde{\epsilon} \cdot P + k \cdot \tilde{\Psi} \tilde{\epsilon} \cdot \tilde{\Psi} \right)$	$3(d - 10 + \frac{m}{6})$
EMS	$V_h, V_\gamma, V_{\tilde{\gamma}}, V_S$ $V_S = (k \cdot \Psi t \cdot \rho) \left( k \cdot \tilde{\Psi} \tilde{\rho} \cdot t \right)$	$3(d - 10 + \frac{m+\tilde{m}}{6})$
BI	$V_{BI} = (k \cdot \Psi_1 k \cdot \Psi_2) \left( \tilde{\epsilon} \cdot P + k \cdot \tilde{\Psi} \tilde{\epsilon} \cdot \tilde{\Psi} \right)$	$\frac{1}{2}(7d - 38)$
Galileon	$V_G = (k \cdot \Psi_1 k \cdot \Psi_2) \left( k \cdot \tilde{\Psi}_1 k \cdot \tilde{\Psi}_2 \right)$	$4d - 8$
DBI	$V_{BI}, V_{S_{BI}}$ $V_{S_{BI}} = (k \cdot \Psi_1 k \cdot \Psi_2) \left( k \cdot \tilde{\Psi} t \cdot \tilde{\rho} \right)$	$\frac{1}{2}(7d + m - 38)$
$T^*$ YM	$V_g = (1/2t \cdot [\rho, \rho]) \left( \tilde{\epsilon} \cdot P + k \cdot \tilde{\Psi} \tilde{\epsilon} \cdot \tilde{\Psi} \right)$ $V_{\tilde{g}} = (t \cdot ([\rho, \tilde{\rho}] + [q, y])) \left( \tilde{\epsilon} \cdot P + k \cdot \tilde{\Psi} \tilde{\epsilon} \cdot \tilde{\Psi} \right)$	$\frac{5}{2}(d - 12)$
$ET^*$ YM	$V_h, V_g, V_{\tilde{g}}$ $V_g = (k \cdot \Psi t \cdot \rho + 1/2t \cdot [\rho, \rho]) \left( \tilde{\epsilon} \cdot P + k \cdot \tilde{\Psi} \tilde{\epsilon} \cdot \tilde{\Psi} \right)$ $V_{\tilde{g}} = (k \cdot \Psi t \cdot \tilde{\rho} + t \cdot ([\rho, \tilde{\rho}] + [q, y])) \left( \tilde{\epsilon} \cdot P + k \cdot \tilde{\Psi} \tilde{\epsilon} \cdot \tilde{\Psi} \right)$	$3(d - 10)$
NLSM	$V = (1/2t \cdot [\rho, \rho]) \left( k \cdot \tilde{\Psi}_1 k \cdot \tilde{\Psi}_2 \right)$ $\tilde{V} = (t \cdot ([\rho, \tilde{\rho}] + [q, y])) \left( k \cdot \tilde{\Psi}_1 k \cdot \tilde{\Psi}_2 \right)$	$3d - 19$

Table 1.4 Table of the different theories and their integrated vertex operators.



## Chapter 2

# Ambitwistor Strings on a Group

While the ambitwistor string has proven very successful for finding new perspectives on amplitudes in flat space, it should in principle be possible to use it to study perturbative QFT even on curved backgrounds. Indeed, the ambitwistor string action was defined in principle on an arbitrary background in [47]. In stark contrast to standard string theory, the action is free and chiral even on an arbitrary manifold. The nontrivial geometry only shows up in the gauge symmetries of the model. These become rather unwieldy for a general background, which has so far hindered significant progress.

The general framework has been successfully used to obtain three-point functions on plane wave geometries [17, 16], but higher point calculations quickly become challenging for various reasons.

Here we propose to study the ambitwistor model on  $AdS_3$ , thought of as a group manifold.

The maximally symmetric  $AdS_{d+1}$  spaces are a natural first step away from flat space. They have a large number of isometries, which allows for immense computational control, and are generally of great interest, not least due to the AdS/CFT duality.

The main advantage of the  $AdS_3$  versus higher dimension  $AdS$  backgrounds is that it can be supported by pure NS flux. As a result, string theory on  $AdS_3 \times S^3$  can be described in the RNS description. Using the isomorphism  $AdS_3 \simeq SL(2, \mathbb{R})$ , string theory on this background can essentially be solved in terms of the representation theory of chiral algebras [94–97].

If instead the background is supported by RR flux,  $AdS_3 \times S^3$  becomes the supergroup  $PSU(1, 1|2)$ , and constructing a string theory requires the Green-Schwarz formalism [98–100] or pure spinors.

The model we propose can be thought of as a change of frame, from the coordinate frame based approach of [47], to a Cartan frame. This non-coordinate frame is defined naturally on a group manifold, and its major advantage is that the metric and the Riemann tensor are *constant*. This is essentially a consequence of the defining relation

$$[e_a, e_b] = f_{ab}^c e_c \quad (2.1)$$

for the frame fields  $e_a$ , which generate the isometries arising from the group left- or right-multiplication. The  $f_{ab}^c$  are the structure *constants* of the group. A relation of this form holds for any frame field, on any manifold, but the coefficients  $f_{ab}^c$  are in general functions of the coordinates. Our model is a rather tractable as a consequence of the  $f_{ab}^c$  being constant and can be studied using well-known representation theory techniques developed for WZW models.

The ambitwistor string in this language has the appearance of a first-order formulation of the WZW model. It is however crucially chiral and does not require an explicit Wess-Zumino term for conformal invariance.

## 2.1 The Worldsheet theory

We begin by describing the classical aspects of our model, before moving on to discuss the consistency conditions that quantisation imposes. The classical analysis is valid for an arbitrary group<sup>1</sup>  $G$ , and can be thought of as a change of variables from the ambitwistor string defined on arbitrary Riemann manifolds, so we keep the notation general and don't use specific properties of  $AdS_3 \times S^3$  until discussing quantization.

The world-sheet action of our model is a generalisation of the ambitwistor action [13, 47], which on a Minkowski background gives rise to the flat-space scattering equations and the CHY framework. The bosonic part of the action is

$$S = \int_{\Sigma} j_a (g^{-1} \bar{\partial} g)^a, \quad (2.2)$$

where  $g$  is a coordinate on the group manifold and  $j$  is valued in the dual of the Lie algebra. Given that the combination  $g^{-1} \bar{\partial} g$  is an element of the algebra, the pairing in the action is canonical and does not require a metric to raise or lower indices. On the worldsheet, the field  $g(z)$  is a scalar, while  $j$  is a holomorphic one-form  $j_{za}(z) dz \hat{t}^a$ ,

$$g \in \Omega(\Sigma, \mathcal{G}), \quad j \in \Omega(\Sigma, K_{\Sigma}) \otimes \mathfrak{g}^*, \quad (2.3)$$

---

<sup>1</sup>which admits a non-degenerate invariant quadratic form on the Lie algebra

making (2.2) the integral of a  $(1, 1)$ -form.

To find the symplectic form, Poisson brackets, Noether currents and equations of motion, we compute the variation of the action

$$\delta S = \int \delta j \cdot (g^{-1} \bar{\partial} g) - (\bar{\partial} j + [g^{-1} \bar{\partial} g, j]) \cdot (g^{-1} \delta g) + \oint j \cdot (g^{-1} \delta g) . \quad (2.4)$$

From this we can read off the classical equations of motion in the absence of sources

$$\bar{\partial} j = 0 , \quad \bar{\partial} g = 0 , \quad (2.5)$$

whose solutions on the sphere are  $j(z) = 0$  and  $g(z) = \text{const.}$  We can further read off the symplectic form

$$\Omega = \oint \delta j \cdot (g^{-1} \delta g) - j \cdot [g^{-1} \delta g, g^{-1} \delta g] \quad (2.6)$$

which allows us to find the Poisson brackets<sup>2</sup>

$$\begin{aligned} \{g, g\} &= 0 , & \{j_a(\sigma), g(\sigma')\} &= \delta(\sigma - \sigma') g t_a , \\ \{j_a(\sigma), j_b(\sigma')\} &= \delta(\sigma - \sigma') f_{ab}^c j_c , \end{aligned} \quad (2.7)$$

where  $t_a$  are a basis of generators of the Lie algebra  $\mathfrak{g}$ , and  $f_{ab}^c$  are the structure constants, with  $[t_a, t_b] = f_{ab}^c t_c$ . This identifies the field  $j$  as Kac-Moody current associated with *right translations* on the group manifold. Indeed, the action (2.2) is invariant under the transformations

$$g(z) \rightarrow h_L g(z) h_R , \quad j(z) \rightarrow h_R^{-1} j(z) h_R , \quad (2.8)$$

with  $h_L, h_R \in \mathcal{G}$ , which describe global left/right translations on the group manifold. The Noether charges of these symmetries are computed to be  $\oint g j g^{-1}$  and  $\oint j$  respectively, and satisfy the expected Poisson bracket relations

$$\begin{aligned} \{j_a(\sigma), j_b(\sigma')\} &= \delta(\sigma - \sigma') f_{ab}^c j_c , & \{(g j g^{-1}), j\} &= 0 , \\ \{(g j g^{-1})_a(\sigma), (g j g^{-1})_b(\sigma')\} &= \delta(\sigma - \sigma') f_{ab}^c (g j g^{-1})_c . \end{aligned} \quad (2.9)$$

We highlight the asymmetry between the left and right translations, which will become ever more prominent as we go along.

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<sup>2</sup>For the sake of brevity we omit the dependence on the string coordinate and the  $\delta$ -function on the right hand side.

It will be important later to understand the action of the right translation generator  $j$  on functions defined on the group manifold. Given some coordinate chart  $x^\mu$  on the group manifold, such that we have  $g = g(x)$ , we can expand

$$g^{-1} \partial_\mu g = e_\mu^a(x) t_a \quad (2.10)$$

where  $e_\mu^a(x)$  is called the left-invariant frame or Maurer-Cartan frame. We can use this to expand the right action on a function  $\varphi(x)$  in terms of derivatives

$$\{j_a, \varphi(x)\} = e_a^\mu \partial_\mu \varphi(x) . \quad (2.11)$$

With this definition it is clear that the vector fields  $e_a \equiv e_a^\mu \partial_\mu$  satisfy the algebra

$$[e_a, e_b] = f_{ab}^c e_c . \quad (2.12)$$

The right translation extends naturally to act on tensor fields defined on the group via the Lie derivative  $\mathcal{L}_a$  along  $e_a$ . We can find the expression for the Lie derivative in the Cartan frame by considering for instance the action on a one-form  $V_\mu dx^\mu \equiv V_a (g^{-1} dg)^a$

$$t_a^{(R)} \left( V_b(g) (g^{-1} dg)^b \right) = (e_a^\mu \partial_\mu V_b - f_{ab}^c V_c) (g^{-1} dg)^b , \quad (2.13)$$

so the components  $V_a$  transform as  $\mathcal{L}_a V_b = e_a(V_b) - f_{ab}^c V_c$ . This extends in the familiar fashion to tensor fields of higher rank.

Using the group structure, we can always find a metric  $m_{ab}$  which is constant and right-invariant

$$\partial_\mu m_{bc} = 0 , \quad m_{ad} f_{bc}^d = f_{ab}^d m_{cd} . \quad (2.14)$$

We will sometimes use the abbreviation  $f_{abc} \equiv m_{ad} f_{bc}^d$ , which is totally antisymmetric as a consequence of (2.14). The inverse/dual frame field  $e_\mu^a$  serves as vielbein for this metric, since the metric in the coordinates  $x^\mu$  is given by

$$ds^2 = m(g^{-1} \partial_\mu g, g^{-1} \partial_\nu g) dx^\mu dx^\nu = m_{ab} e_\mu^a e_\nu^b dx^\mu dx^\nu . \quad (2.15)$$

One possible choice for an invariant metric is the Killing form  $\kappa_{ab} \equiv f_{ac}^d f_{bd}^c$ , but we emphasize that semi-simple groups may admit several invariant quadratic forms. Indeed, the group underlying  $AdS_3 \times S^3$  has two independent quadratic invariant forms, and we will show below that quantum consistency of the model fixes a unique choice.

Given the metric  $m$  we can construct its Levi-Civita connection  $\nabla_\mu$ , or  $\nabla_a \equiv e_a^\mu \nabla_\mu$ , which acts in the usual fashion on tensor fields on the group manifold. For the



connection to act on tensors carrying algebra indices, we need the spin connection  $\omega_{\mu a}^b$ , for example on the metric  $m$ :

$$\nabla_\mu m_{ab} = \partial_\mu m_{ab} - \omega_{\mu a}^c m_{cb} - \omega_{\mu b}^c m_{ac}$$

and similarly for tensors of other rank. The spin connection is defined by the equation

$$\nabla_\mu e_\nu^a \equiv \partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\rho e_\rho^a + \omega_{\mu b}^a e_\nu^b = 0 \quad (2.16)$$

which can be solved for  $\omega$  as

$$\omega_{\mu a}^b = -\frac{1}{2} f_{ac}^b e_\mu^c . \quad (2.17)$$

Notice that this relation between the spin connection and the structure constants means the invariance conditions (2.14) and  $\nabla_a m_{bc} = 0$  are compatible.

We will need these tools later to show that the BRST cohomology of our model consists of solutions to the linearised supergravity equations on the appropriate background.

The action (2.2) has a further symmetry, which plays a crucial role in our construction. Under this symmetry, the fields transform as

$$\delta g = \alpha (gt_a) m^{ab} j_b , \quad \delta j = 0 , \quad (2.18)$$

where  $\alpha(z)$  is an arbitrary holomorphic vector field parametrizing the transformation and  $m^{ab}$  is the inverse of the metric  $m_{ab}$ . This symmetry transformation is generated by  $\oint \alpha H$ , where

$$H = \frac{1}{2} m^{ab} j_a j_b . \quad (2.19)$$

We are going to gauge this symmetry, and thus introduce a Lagrange multiplier field  $\tilde{e}$  and extend the action to

$$S = \int_\Sigma j \cdot (g^{-1} \bar{\partial} g) + \tilde{e} H . \quad (2.20)$$

To make the integrand a  $(1,1)$ -form, and thus well defined, the field  $\tilde{e}$  has to carry the conformal weight of a Beltrami differential,  $\tilde{e} = \tilde{e}_z^{\bar{z}} d\bar{z} \otimes \partial_z \in \Omega^1(\Sigma, T_\Sigma)$ . Beltrami differentials arise as the infinitesimal deformations of the complex structure on  $\Sigma$ , and span the tangent space to the space of complex structures at the point  $\Sigma$ . The field  $\tilde{e}$  is therefore best understood a coordinate on the cotangent space to the moduli space [14].

The action of the gauge symmetry is now

$$\delta\tilde{e} = -\bar{\partial}\alpha, \quad \delta g = \alpha(gt_a)m^{ab}j_b, \quad \delta j = 0, \quad (2.21)$$

where  $\alpha$  is any smooth (not necessarily holomorphic) vectorfield. In the flat space ambitwistor model of [13], it is the gauging this current  $H$  which eventually leads to a localisation of correlation functions onto the solutions of some algebraic equations, the scattering equations. It is an exciting question whether this localisation persists on AdS.

Due to the chiral nature of the action, we anticipate that the bosonic model as it stands is not diffeomorphism invariant at the quantum level. We will describe below how the partition function gives rise to this anomaly, which is also present in other ambitwistor models [13, 47, 31], and the problems it causes. Fortunately the issue can be resolved by introducing a pair of ‘real’ fermions  $\psi, \tilde{\psi}$  with the action

$$S_\psi = \int_\Sigma \frac{1}{2} m(\psi, \bar{\partial}\psi + [g^{-1}\bar{\partial}g, \psi]) = \int_\Sigma \frac{1}{2} m_{ab} \psi^a (\bar{\partial}\psi^b + f_{cd}^b (g^{-1}\bar{\partial}g)^c \psi^d) \quad (2.22)$$

and likewise for  $\tilde{\psi}$ . The fields  $\psi, \tilde{\psi}$  carry a Lie algebra index, and are spin-1/2 on the world-sheet

$$\psi, \tilde{\psi} \in \Omega(\Sigma, K_\Sigma^{1/2}) \otimes \mathfrak{g}. \quad (2.23)$$

In components this means  $\psi(z) = \psi_z^a(z) \sqrt{dz} t_a$ , and likewise for  $\tilde{\psi}$ . Their partition function, cancels the partition function of the bosonic fields, and thus the anomalous behaviour is cured. The non-zero Poisson brackets involving the fermions are given by

$$\{j_a(\sigma), \psi^b(\sigma')\} = -\delta(\sigma - \sigma') f_{ac}^b \psi^c, \quad \{\psi^a(\sigma), \psi^b(\sigma')\} = m^{ab} \delta(\sigma - \sigma'), \quad (2.24)$$

and likewise for  $\tilde{\psi}$ . This shows that the fermions transform in the standard<sup>3</sup> way under the right-translation Kac-Moody algebra generated by  $j_a$ , and are inert under left-translations.

Our proposal for the ambitwistor string describing supergravity on  $AdS_3 \times S^3$  is the combined model (2.2)+(2.22) with the action

$$S = \int_\Sigma j \cdot (g^{-1}\bar{\partial}g) + \frac{1}{2} m(\psi, \bar{\partial}\psi + [g^{-1}\bar{\partial}g, \psi]) + \frac{1}{2} m(\tilde{\psi}, \bar{\partial}\tilde{\psi} + [g^{-1}\bar{\partial}g, \tilde{\psi}]) \quad (2.25)$$

and after gauging various symmetries, which we describe next.

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<sup>3</sup>We can therefore build invariants simply by contracting indices, e.g.  $f_{abc} \psi^a \psi^b \psi^c$ .

Having included the two fermions we need to revisit the various symmetries of the model. Target space right-translations are still generated by the field  $j_a(z)$ , while left-translations are generated by the composite operator

$$j_L = g \left( j - \frac{1}{2} \psi^2 - \frac{1}{2} \tilde{\psi}^2 \right) g^{-1} . \quad (2.26)$$

Furthermore, the combined model has a two fermionic symmetries, generated by the currents<sup>4</sup>

$$\begin{aligned} G &= j \cdot \psi + \frac{1}{2} \mathfrak{m}(\psi, [\tilde{\psi}, \tilde{\psi}]) + \frac{1}{6} \mathfrak{m}(\psi, [\psi, \psi]) \\ \tilde{G} &= j \cdot \tilde{\psi} + \frac{1}{3} \mathfrak{m}(\tilde{\psi}, [\tilde{\psi}, \tilde{\psi}]) . \end{aligned} \quad (2.27a)$$

The current  $H$  (2.19) is modified:

$$H = \frac{\mathfrak{m}^{ab}}{2} \left( j_a + \frac{1}{2} \mathfrak{f}_{acd} \tilde{\psi}^c \tilde{\psi}^d \right) \left( j_b + \frac{1}{2} \mathfrak{f}_{bef} \tilde{\psi}^e \tilde{\psi}^f \right) \quad (2.27b)$$

and together the currents  $G, \tilde{G}, H$  satisfy a  $SL(1|2)$  super-algebra

$$\begin{aligned} \{G, G\} &= 2H , & \{\tilde{G}, \tilde{G}\} &= 2H , & \{G, \tilde{G}\} &= 0 , \\ \{G, H\} &= 0 , & \{\tilde{G}, H\} &= 0 \end{aligned} \quad (2.27c)$$

of symmetries, which we now proceed to gauge. Gauging  $G, \tilde{G}$  requires two fermionic Lagrange multipliers  $\chi, \tilde{\chi} \in \Omega^1(\Sigma, T_\Sigma^{1/2})$ , in addition to the bosonic Lagrange multiplier  $\tilde{e} \in \Omega^1(\Sigma, T_\Sigma)$  introduced for gauging  $H$ . Concretely, we add to the action (2.25) the terms  $\int \tilde{e} H + \chi G + \tilde{\chi} \tilde{G}$ , and extend the action of the gauge symmetries on the Lagrange multipliers in the familiar fashion, such that

$$\begin{aligned} \delta_{\tilde{e}}(\tilde{e}, \chi, \tilde{\chi}) &= (-\bar{\partial}\tilde{e}, 0, 0) , & \delta_\gamma(\tilde{e}, \chi, \tilde{\chi}) &= (-2\chi\gamma, -\bar{\partial}\gamma, 0) , \\ \delta_{\tilde{\gamma}}(\tilde{e}, \chi, \tilde{\chi}) &= (-2\tilde{\chi}\tilde{\gamma}, 0, -\bar{\partial}\tilde{\gamma}) , \end{aligned} \quad (2.28)$$

to make the full action

$$S = S_{j,g} + S_\psi + S_{\tilde{\psi}} + \int \tilde{e} H + \chi G + \tilde{\chi} \tilde{G} \quad (2.29)$$

gauge invariant.

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<sup>4</sup>classically the model has a  $\mathbb{C}$ -parameter family of symmetries, but demanding the charges to close into a  $SL(2)$  algebra quantum mechanically selects one particular value.

### 2.1.1 Quantization

We begin by discussing the quantization of the free system, adding the gauge constraints later via BRST quantization. We first demonstrate that the OPEs of (2.25) are given by

$$\begin{aligned}
 j_a(z) g(w) &\sim \frac{1}{z-w} g(w) t_a, & j_a(z) j_b(w) &\sim \frac{1}{z-w} f_{ab}^c j_c(w), \\
 j_a(z) \psi^b(w) &\sim \frac{-1}{z-w} f_{ac}^b \psi^c, & j_a(z) \tilde{\psi}^b(w) &\sim \frac{-1}{z-w} f_{ac}^b \tilde{\psi}^c, \\
 \psi^a(z) \psi^b(w) &\sim \frac{1}{z-w} m^{ab}, & \tilde{\psi}^a(z) \tilde{\psi}^b(w) &\sim \frac{1}{z-w} m^{ab},
 \end{aligned} \tag{2.30}$$

with all others vanishing. Notice the absence of a level in the  $j$ -Kac-Moody algebra. The OPEs with the field  $j_a(z)$  can be derived using the Ward identity of right translations [101]. Consider right-translation by a general, smooth, group-valued function  $h(z, \bar{z})$ , under which the action behaves as

$$S[g h, h^{-1} j h, h^{-1} \psi h, h^{-1} \tilde{\psi} h] = S[j, g, \psi, \tilde{\psi}] + \int j \cdot (\bar{\partial} h h^{-1}). \tag{2.31}$$

To derive the Ward identity we have to understand the transformation of the path integral measure, or equivalently, the partition function. The partition function of the bosonic theory,  $S[j, g]$  (2.2), is given by a functional determinant of a chiral  $\bar{\partial}$ -operator. The gauge anomalies of these operators have been studied in great detail; see e.g. for a review [102]. Its variation under an infinitesimal change of the connection  $\bar{\partial} \rightarrow \bar{\partial} + \theta$  is given by the Quillen construction [103] of the determinant line bundle

$$\delta_\theta \ln \det(\bar{\partial}) \propto \left\langle \int_\Sigma f_{ab}^c (g^{-1} \partial g)^a \theta_c^b \right\rangle. \tag{2.32}$$

This leads in particular to an anomalous behaviour of the quantum partition function under target space diffeomorphisms, which in turn inflicts all correlation functions based on this vacuum. Consequently, attempting to perform BRST quantization creates a host of problems. For instance, the resulting vertex operators don't satisfy the desired supergravity equations of motion, and the Kac-Moody algebra acquires a level of  $-1$ . The full model (2.25) however has equal numbers of bosonic and fermionic degrees of freedom, so the functional determinant cancels out of the partition function. Consequently this indicates that the path integral measure is indeed invariant, and the  $j$ -Kac-Moody algebra has level zero.

Taking  $h = \exp \xi$  and expanding to first order leads to the Ward identity

$$\left\langle \left( \int j \cdot \bar{\partial} \xi \right) \prod_i \mathcal{O}_i(z_i) \right\rangle = \sum_i \left\langle (\xi \cdot \delta \mathcal{O}_i)(z_i) \prod_{\ell \neq i} \mathcal{O}_\ell(z_\ell) \right\rangle \quad (2.33)$$

where the operators  $\mathcal{O}_i$  can be any of the fundamental fields  $j, g, \psi, \tilde{\psi}$  and  $\delta \mathcal{O}_i$  is the first order variation under right-translations. Specializing the algebra valued function  $\xi$  to be of the form

$$\xi(z) = t_a \frac{1}{z - z_0} , \quad (2.34)$$

for some fixed algebra element  $t_a$  and some arbitrary puncture location  $z_0$  yields

$$\left\langle j_a(z_*) \prod_i \mathcal{O}_i(z_i) \right\rangle = \sum_i \frac{dz_*}{z_* - z_i} \left\langle (t_a \cdot \delta \mathcal{O}_i)(z_i) \prod_{\ell \neq i} \mathcal{O}_\ell(z_\ell) \right\rangle . \quad (2.35)$$

From this the OPEs (2.30) of  $j_a$  with the fundamental fields are easily read off by picking out the appropriate terms. Notice that regularity of this correlator at  $z_* \rightarrow \infty$  requires the right hand side to fall off as  $z_*^{-2}$ . which is the case iff

$$0 = \sum_i \left\langle (t_a \cdot \delta \mathcal{O}_i)(z_i) \prod_{\ell \neq i} \mathcal{O}_\ell(z_\ell) \right\rangle . \quad (2.36)$$

This requirement is tantamount to the statement that any correlator computed in this theory is invariant under global right-multiplication. (An insertion of the left-multiplication current similarly yields invariance under left-multiplications.) This is analogous to ambitwistor string on flat space, global translation invariance gives rise to an momentum conserving  $\delta$ -function via the zero mode of the field  $X(z)$ , which in turn guarantees the absence of spurious poles in the field  $P(z)$ .

We emphasise that eq. (2.35) is an exact statement, not merely the leading parts of an expansion or approximation. Moreover, it is worth pointing out that this derivation used here only works so straightforwardly because of the absence of normal ordering issues, which may introduce higher order poles into the OPE of  $j$  with composite operators such as  $f_{bc}^a \psi^b \psi^c$ . We will illustrate how such higher order OPEs can be derived using the Ward identity method together with point-splitting below.

The field  $g$  has no nontrivial OPEs other than with  $j$ , i.e.

$$g(z) g(w) \sim 0 , \quad g(z) \psi^a(w) \sim 0 , \quad g(z) \tilde{\psi}^a(w) \sim 0 . \quad (2.37)$$

This can be shown e.g. by computing the corresponding two-point functions. In the absence of any  $j$ -field insertion, the field  $j$  can be integrated out, which localizes the remaining correlator onto field configurations where  $g(z)$  is holomorphic, and thus constant  $g(z) = g_0 \in \mathcal{G}$ . This means the correlator is regular everywhere as a function of the locations of  $g$ , and thus the OPEs of  $g$  vanish. Thus the most elementary correlators

$$\langle \varphi_1(g(z_1)) \varphi_2(g(z_2)) \cdots \varphi_n(g(z_n)) \rangle \quad (2.38)$$

of a product of functions on the group, potentially in various representations  $\mathcal{R}$ , localize to the finite dimensional group integral

$$\int_{\mathcal{G}} d^d g_0 \varphi_1(g_0) \varphi_2(g_0) \cdots \varphi_n(g_0) \quad (2.39)$$

over the zero modes. In other words, the correlation function reduces to the finite dimensional group integral and the dependence on the locations  $z_i$  simply drops out.

Looking ahead to the case of  $AdS_3$ , we will encounter (derivatives of) correlators of the form

$$\langle \text{tr}(h_1 g(z_1))^{-\Delta_1} \cdots \text{tr}(h_n g(z_n))^{-\Delta_n} \rangle = \int_{AdS_3} d^3 g_0 \prod_i \frac{1}{\text{tr}(h_i g_0)^{\Delta_i}} \quad (2.40)$$

where the  $h_i$  are some fixed matrices, with vanishing determinant, representing boundary points. These are then nothing but the familiar  $n$ -point scalar contact interactions, the so called D-functions [104]. We discuss this integral in more detail below.

We point out that the absence of  $g g$  OPEs makes our model dramatically easier to work with than the standard string on a group manifold, e.g. regarding the construction of vertex operators. This absence is a characteristic feature of ambitwistor string models, which describe target space *field theories* despite their stringy setup.

The remaining OPEs are the ones among the fermions. The most useful way to obtain these is to compute the fermion propagator, defined as the solution  $G^{ab}(w, z)$  to the equation

$$\left( \delta_b^a \bar{\partial} + f_{bc}^a (g^{-1} \bar{\partial} g)^c \right) G^{bd}(w, z) = \bar{\delta}(w - z) m^{ad}. \quad (2.41)$$

Note that this equation holds for arbitrary, fixed  $g(z)$ . The solution can be found by conjugation from the ‘free’ fermion propagator and reads

$$G^{ab}(w, z) = \frac{\sqrt{dw} \sqrt{dz}}{w - z} m^{-1} \left( g(w) t^a g^{-1}(w), g(z) t^b g^{-1}(z) \right), \quad (2.42)$$

where the  $t^a$  are a dual basis of the algebra. Expanding for small  $w - z$ , and using the invariance of the metric, yields the standard free fermion propagator.

With the fundamental OPEs established we can compute the quantum corrections to the  $SL(1|2)$  algebra of gauge symmetries (2.27). While straightforward in principle, the computation is lengthy and rather cumbersome at intermediary stages, because the normal ordering or point-splitting of composite operators has to be kept track of carefully. As an example of the normal ordering subtleties that arise, consider the OPE of  $j_a(z)$  with the operator  $\psi^3(w) \equiv f_{abc} \psi^a \psi^b \psi^c(w)$ . The classical Poisson bracket, and thus the simple pole in the OPE, is given by the infinitesimal variation under right translations of  $\psi^3$ . Classically, it is enough to observe that  $\psi^3(w)$  has no free indices to conclude that it is invariant, and therefore

$$\{j_a(z), \psi^3(w)\} = 0 . \quad (2.43)$$

However, to define the composite operator quantum mechanically we need to point-split. Generically, point-splitting may introduce ambiguities in the definition, such that different ways of point-splitting lead to different operators after removing the regulator. This may produce corrections to (2.43) at the quantum level. In fact, the operator  $\psi^3$  is itself free of such ambiguities, since the definition

$$\psi^3(w) := \lim_{\varepsilon \rightarrow 0} f_{abc} \psi^a(w - \varepsilon) \psi^b(w) \psi^c(w + \varepsilon) \quad (2.44)$$

is well-defined and finite as  $\varepsilon \rightarrow 0$ , because  $f_{abc} m^{bc} = 0$ . Because this product of operators is actually finite, any other way of point-splitting will land on the same operator after removing the regulator. Nonetheless, performing the OPE of  $j_a(z)$  with any point-split definition of  $\psi^3$  we find

$$j_a(z) \psi^3(w) \sim \frac{-3\kappa_{ab}}{(z-w)^2} \psi^b(w) . \quad (2.45)$$

This agrees with the classical (vanishing) result at the first order pole, but has a quantum mechanical second order pole proportional to the Killing form.

Another example where a quantum mechanical subtlety arises is the operator

$$f_{ab}^e f_{cde} \psi^a \psi^b \psi^c \psi^d . \quad (2.46)$$

Classically, this combination would vanish identically as a consequence of the Jacobi identity, since the fermions effect a total anti-symmetrization of the indices. Quantum

mechanically, we have to define this composite operator via point splitting, which precludes the anti-symmetrization, so the Jacobi identity ceases to apply. We might for example try to define this operator as

$$\lim_{\varepsilon \rightarrow 0} f_{ab}^e f_{cde} \psi^a(z - 2\varepsilon) \psi^b(z - \varepsilon) \psi^c(z + \varepsilon) \psi^d(z + 2\varepsilon) \quad (2.47)$$

but this limit is singular. Therefore we are left with no choice but to subtract the singularity, which is of the form  $\frac{1}{\varepsilon} \kappa_{ab} \psi^a \psi^b$  and  $\frac{1}{\varepsilon^2} \kappa_{ab} m^{ab}$ . There is no canonical way of performing this subtraction, and closer inspection reveals that the various different choices end up differing by a multiple of the operator

$$\kappa_{ab} : \psi^a \partial \psi^b : \quad (2.48)$$

after the regulator is removed.<sup>5</sup>

We emphasise that the currents  $G, \tilde{G}, H$  defined in (2.27) are each free of normal ordering ambiguities, since all potential ambiguities vanish. Equivalently, they can be defined via point-splitting, and the limit of removing the regulator is actually finite without any further subtractions. This does however not preclude quantum effects from changing the OPEs between them. Indeed, while the operators  $G, \tilde{G}, H$  themselves behave naturally under quantisation, it is not guaranteed that the symmetries they generate, or the algebra structure (2.27) survive.

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<sup>5</sup>The equivalent statement in the canonical framework is that the definition of this composite operator requires some ordering prescription. In terms of the standard normal ordering  $: \cdot :$ , we could define the operator for instance as

$$f_{ab}^e f_{cde} : (\psi^a \psi^b) : (\psi^c \psi^d) : \quad \text{or} \quad f_{ab}^e f_{cde} : (\psi^a \psi^b) : \psi^c : \psi^d : ,$$

and, again, these definitions differ by a multiple of  $\kappa_{ab} : \psi^a \partial \psi^b :$ . In this special case, since the classical limit of any of these operators is zero by the Jacobi identity, the operators can be reduced entirely to the normal ordering ambiguity, for example

$$f_{ab}^e f_{cde} : (\psi^a \psi^b) : (\psi^c \psi^d) : = -4 \kappa_{ab} : \psi^a \partial \psi^b :$$

holds as an operator equation. The right hand side is a purely quantum contribution, arising from a double contraction.



Explicit computations<sup>6</sup> of the OPEs between  $G, \tilde{G}, H$  yield

$$\begin{aligned} G(z) G(w) &\sim -\frac{1}{3} \frac{\kappa_{ab} m^{ab}}{(z-w)^3} + \frac{2H}{z-w} , \\ G(z) \tilde{G}(w) &\sim 0 , \\ \tilde{G}(z) \tilde{G}(w) &\sim -\frac{1}{3} \frac{\kappa_{ab} m^{ab}}{(z-w)^3} + \frac{2H}{z-w} . \end{aligned} \quad (2.49)$$

The presence of the cubic poles potentially violates the gauge algebra. The anomaly can be made to vanish if there is a metric  $m$  that obeys

$$\kappa_{ab} m^{ab} = 0 . \quad (2.50)$$

In other words, we require that the group manifold  $\mathcal{G}$  admits a bi-invariant metric  $m$ , such that the  $m$  trace of the Killing form vanishes. Standard WZW models on such group manifolds have been classified and extensively studied, see e.g. [106–112].

At this point we choose to study the case  $\mathcal{G} = SO(4) \simeq AdS_3 \times S^3$ . To understand the meaning of the constraint  $\kappa_{ab} m^{ab} = 0$  in this context we build on the intuition of [47], where it was shown that the survival of the ambitwistor string  $SL(1|2)$  world-sheet algebra at the quantum level imposes conditions on the geometry of the background space-time. More precisely, the  $SL(1|2)$  algebra is consistent at the quantum level *iff* the background satisfies the type II supergravity equations

$$R_{\mu\nu} - \frac{1}{4} H_{\mu\kappa\lambda} H_{\nu}{}^{\kappa\lambda} + 2 \nabla_{\mu} \nabla_{\nu} \Phi = 0 , \quad (2.51a)$$

$$\nabla_{\kappa} H^{\kappa}{}_{\mu\nu} - 2 H^{\kappa}{}_{\mu\nu} \nabla_{\kappa} \Phi = 0 , \quad (2.51b)$$

$$R + 4 \nabla_{\mu} \nabla^{\mu} \Phi - 4 \nabla_{\mu} \Phi \nabla^{\mu} \Phi - \frac{1}{12} H^2 = 0 . \quad (2.51c)$$

Here  $R_{\mu\nu}$  is the Ricci tensor,  $R$  the Ricci scalar,  $H_{\mu\nu\kappa}$  the NS three-form and  $\Phi$  the dilaton respectively.

Since our model is defined on a group manifold, whose geometry is highly constrained, it is already a long way towards being a consistent background. Restricted to a group manifold, the differential equations (2.51) turn into algebraic ones [113]. Using the relation between the connection and the structure constants (2.17) as well as the

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<sup>6</sup>The computation has been performed with the help of the MATHEMATICA software package Lambda [105].

right-invariance of the metric, it is simple to show that the Riemann tensor is constant

$$R_{abc}{}^d = \frac{1}{4} f_{ab}^e f_{ec}^d, \quad (2.52)$$

e.g. by evaluating  $[\nabla_a, \nabla_b] V_c \equiv R_{abc}{}^d V_d$ . Consequently, the Ricci tensor is given by the Killing form,  $R_{ab} = -\frac{1}{4} \kappa_{ab}$ . On a simple Lie group, such as  $AdS_3$ , the Killing form is the unique invariant quadratic form, and thus the metric is necessarily a multiple of the Killing form. This means that the bosonic Einstein equation is automatically satisfied, with some suitable value for the cosmological constant. On a semi-simple group, however, such as  $AdS_3 \times S^3$ , it is possible to choose a metric different from the Killing form. In fact, the equations (2.51) do not include an explicit cosmological constant, but instead use the NS three-form flux to stabilise the geometry.

Given that the  $AdS_3 \times S^3$  dilaton is constant in the Cartan frame, we can deduce from eqs. (2.51a) and (2.51b) the identification

$$H_{\mu\nu\kappa} = -e_\mu^a e_\nu^b e_\kappa^c f_{abc} \quad (2.53)$$

for the background NS three-form. Using these identifications, the dilaton equation (2.51c) becomes

$$-\frac{1}{3} \kappa_{ab} m^{ab} = 0 \quad (2.54)$$

which is precisely the ‘anomalous’ term in the OPEs (2.49), as expected from eq. (2.51).

In the case of  $AdS_3 \times S^3$  it is simple to solve (2.54) explicitly (see e.g. [106, 99]). The solution is most easily understood with the help of the isomorphism  $\text{Lie}(AdS_3 \times S^3) \simeq \mathfrak{so}(4)$ . This isomorphism introduces an auxiliary  $\mathbb{R}^{1,3}$ , with a flat Minkowski metric  $\eta_{mn}$ . Every Lie algebra index is exchanged for an antisymmetric pair of auxiliary  $\mathbb{R}^{1,3}$  indices, i.e.  $t_a \simeq t_{mn}$  with  $t_{mn} = -t_{nm}$ . A straightforward computation shows that the Killing form is given by

$$\kappa_{ab} \simeq \eta_{mp} \eta_{nq} - \eta_{mq} \eta_{np}, \quad \text{with } a \simeq [mn], \ b \simeq [pq]. \quad (2.55)$$

Since  $\mathfrak{so}(4)$  is semi-simple, with two simple factors, there is a two-dimensional family of quadratic forms, spanned by  $\kappa_{ab}$  and the Levi-Civita symbol  $\varepsilon_{mnpq}$ . Any linear combination of these two tensors is a candidate for the background metric, but the dilaton equation (2.51c)/(2.54) singles out the unique choice

$$m_{ab} \simeq \varepsilon_{mnpq}, \quad \text{with } a \simeq [mn], \ b \simeq [pq], \quad (2.56)$$

up to an overall scale. This structure is well known and has been used extensively to study conventional strings on  $AdS_3 \times S^3$ .

### 2.1.2 The Virasoro algebra

Our model is classically conformally invariant, with the conserved current being the stress tensor

$$T = j_a (g^{-1} \partial g)^a - \frac{\mathbf{m}_{ab}}{2} \psi^a \left( \partial \psi^b + \mathbf{f}_{cd}^b (g^{-1} \partial g)^c \psi^d \right) - \frac{\mathbf{m}_{ab}}{2} \tilde{\psi}^a \left( \partial \tilde{\psi}^b + \mathbf{f}_{cd}^b (g^{-1} \partial g)^c \tilde{\psi}^d \right) \quad (2.57)$$

in the matter sector. Carefully computing the quantum Virasoro algebra, for a generic group manifold of dimension  $d$ , using the OPEs (2.30) we find

$$T(z)T(w) \sim \frac{1}{2} \frac{3d}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial T(w) , \quad (2.58)$$

so we read off the central charge  $c = 3d$ . After BRST quantization the stress tensor receives a further contribution from the ghost sector, which shifts the central charge by  $-26$  for each  $bc$  ghost pair and  $+11$  for each  $\beta\gamma$  ghost pair. Altogether this yields

$$c = 3d - 26 - 26 + 11 + 11 = 3(d - 10) , \quad (2.59)$$

as expected for type II supergravity. Since the group background  $AdS_3 \times S^3$  only makes up for 6 of the total number of dimensions, consistency of the ambitwistor string at higher genus requires that we include a further chiral CFT of  $c = 12$ . The simplest possibility is a chiral CFT describing an internal Ricci flat four-manifold  $X$ , so that the target space becomes  $AdS_3 \times S^3 \times X$ . Here we focus on the states that are independent of the internal CFT.

### 2.1.3 Field redefinition

The form of the gauge constraints  $G, \tilde{G}, H$  (2.27) is suggestive of natural redefinition of the field  $j_a$  as

$$J_a := j_a + \frac{1}{2} \mathbf{f}_{abc} \tilde{\psi}^b \tilde{\psi}^c , \quad (2.60)$$

in terms of which the currents read

$$\begin{aligned} G &= J \cdot \psi + \frac{1}{6} \mathfrak{m}(\psi, [\psi, \psi]) , & \tilde{G} &= J \cdot \tilde{\psi} - \frac{1}{6} \mathfrak{m}(\tilde{\psi}, [\tilde{\psi}, \tilde{\psi}]) , \\ H &= \frac{1}{2} \mathfrak{m}^{-1}(J, J) = \frac{\mathfrak{m}^{ab}}{2} J_a J_b . \end{aligned} \quad (2.61)$$

These currents appear simpler and more symmetric between  $\psi$  and  $\tilde{\psi}$ , while the change from  $j$  to  $J$  has no effect on the algebra (2.49) they satisfy. After this redefinition, the asymmetry is shifted to the OPEs, which now read

$$\begin{aligned} J_a(z) g(w) &\sim \frac{1}{z-w} g(w) t_a , & J_a(z) J_b(w) &\sim \frac{-\frac{1}{2} \kappa_{ab}}{(z-w)^2} + \frac{1}{z-w} f_{ab}^c J_c , \\ J_a(z) \psi^b(w) &\sim \frac{-1}{z-w} f_{ac}^b \psi^c , & J_a(z) \tilde{\psi}^b(w) &\sim 0 , \\ \psi^a(z) \psi^b(w) &\sim \frac{1}{z-w} \mathfrak{m}^{ab} , & \tilde{\psi}^a(z) \tilde{\psi}^b(w) &\sim \frac{1}{z-w} \mathfrak{m}^{ab} . \end{aligned} \quad (2.62)$$

Notice the shifted level in the Kac-Moody algebra, which can be thought of as quantum anomaly in the change of variables coming from the measure. We stress that (taking into account the shifted level) all results computed in either system are identical.

From here on we will only work with the shifted system, and for convenience adopt the notation  $j_a$  for the shifted current  $J_a$ .

## 2.2 Vertex Operators

Following standard BRST quantization procedure, we introduce a ghost anti-ghost pair for each gauge symmetry and add to the action the ghost terms

$$S_{\text{gh}} = \int_{\Sigma} b \bar{\partial} c + \tilde{b} \bar{\partial} \tilde{c} + \beta \bar{\partial} \gamma + \tilde{\beta} \bar{\partial} \tilde{\gamma} \quad (2.63)$$

where the ghosts carry the quantum numbers  $c, \tilde{c} \in \Pi\Omega^0(T_{\Sigma})$  and  $\gamma, \tilde{\gamma} \in \Pi\Omega^0(T_{\Sigma}^{1/2})$ . The anti-ghosts  $b, \tilde{b}, \beta, \tilde{\beta}$  carry the same quantum numbers as the symmetry currents they are gauging, but have opposite statistics. Then we define the BRST operator

$$Q = \oint c T + \tilde{c} H + \gamma G + \tilde{\gamma} \tilde{G} - \tilde{b} \gamma^2 - \tilde{\tilde{b}} \tilde{\gamma}^2 \quad (2.64)$$

and only allow vertex operators which are in the BRST cohomology of  $Q$ . The various symmetries, including conformal invariance, dictate that the fixed vertex operators

take the form

$$U = c \tilde{c} \delta(\gamma) \delta(\tilde{\gamma}) \psi^a \tilde{\psi}^b V_{ab}(g) , \quad (2.65)$$

where  $V_{ab}$  is a world-sheet scalar tensor field on the group. We have imposed the standard type II GSO projection<sup>7</sup> by gauging the discrete  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry given by  $\psi \rightarrow -\psi$  and  $\tilde{\psi} \rightarrow -\tilde{\psi}$ . The vertex operator is automatically of conformal weight zero, due to the absence of a  $gg$  OPE. There are non-trivial OPEs between the vertex operator and the  $SL(1|2)$  charges  $G, \tilde{G}, H$  (2.61). For  $U$  to be BRST closed, we require that there are no double poles in these OPEs, which imposes the constraints

$$m^{ab} e_a V_{bc} = 0 , \quad m^{ac} (e_a V_{bc} - f_{ab}^d V_{cd}) = 0 , \quad (2.66a)$$

$$m^{ab} (e_a e_b V_{cd} - 2 f_{ac}^e e_b V_{ed} + \kappa_{ac} V_{bd}) = 0 , \quad (2.66b)$$

on the tensor field  $V_{ab}$ . Here,  $e_a \equiv e_a^\mu \partial_\mu$  acts on  $V_{ab}$  component-wise (as opposed to via the Lie derivative). These constraints may be translated into a more familiar form, in terms of the Levi-Civita connection  $\nabla_a \equiv e_a^\mu \nabla_\mu$ , with the help of the relation (2.17) between the structure constants and the connection. Decomposing the tensor  $V_{ab}$  into graviton ( $\delta g$ , symmetric), NS B-field ( $\delta b$ , anti-symmetric) and dilaton ( $\delta \phi$ , scalar) fluctuations as

$$V_{ab} = \delta g_{ab} + \delta b_{ab} + m_{ab} \delta \phi , \quad (2.67)$$

we recognize the first order equations (2.66a) as the de Donder gauge condition in the presence of a background NS three-form *iff* we restrict the fluctuations to satisfy

$$m^{ab} \delta g_{ab} = 4 \delta \phi . \quad (2.68)$$

This condition is actually natural in the context of first order string theory [102, 115], where the dilaton coupling is known to be shifted by  $\phi \rightarrow \phi - \frac{1}{2} \log \sqrt{g}$ . This shift is responsible for turning the usual trace-free condition of the graviton irrep,  $m^{ab} \delta g_{ab} = 0$ , into the condition (2.68). (Alternatively, we could expand  $V$  in terms of the trace-free graviton, B-field and dilaton, but at the expense of an unnatural looking normalisation factors.)

The second order equation (2.66b) decomposes into a symmetric and anti-symmetric part, and is equivalent to the linearization of the supergravity equations (2.51a), (2.51b) and (2.51c). This shows that the BRST physical states of our model indeed encode on-shell supergravity fluctuations around the  $AdS_3 \times S^3$  background. We emphasise that, as in the ambitwistor string around flat space-time, this is an exact statement

<sup>7</sup>For a recent treatment of the GSO projection in the ambitwistor string see [114].

rather than a perturbative one: unlike the standard string, the ambitwistor string has no  $\alpha'$  corrections.

A natural basis for the space of solutions to the linearised supergravity equations (2.66) consists of the so-called bulk-to-boundary propagators. They have been used to construct vertex operators in previous studies of strings on  $AdS_3$  [116, 117]. We describe their construction for the  $AdS_3$  factor, with the understanding that this is only a subsector of the possible solutions on  $AdS_3 \times S^3$ . The bulk-to-boundary propagators are solutions of the equations of motion (without source term), with the special property that they asymptote to a Dirac  $\delta$ -function profile on the boundary of  $AdS$  [118, 94, 119]. The space  $AdS_3$  is isomorphic to the group manifold  $SL(2)$  with appropriate reality conditions on the group encoding on the signature in space-time. Since the ambitwistor string naturally lives in complexified space-time, we won't fix a reality condition.

The space  $AdS_3$  can be expressed in terms of coordinates  $\gamma, \bar{\gamma}, \phi$ , where the metric reads

$$ds^2 = d\phi^2 + e^{2\phi} d\gamma d\bar{\gamma} . \quad (2.69)$$

We have set to unity the  $AdS$  scale, which sets the scale of the constant curvature. The identification with  $SL(2)$  is provided by

$$g(\phi, \gamma, \bar{\gamma}) = e^\phi \begin{pmatrix} \gamma \bar{\gamma} + e^{-2\phi} & \bar{\gamma} \\ \gamma & 1 \end{pmatrix} \in SL(2) . \quad (2.70)$$

This satisfies  $\det g = 1$  identically, and the Cartan frame can be computed explicitly via the definition  $e_\mu^a = (g^{-1} \partial_\mu g)^a$ .

The boundary of  $AdS_3$  corresponds to  $\phi \rightarrow \infty$ . After discarding an infinite overall constant, eq. (2.70) shows that the boundary is parametrized by matrices of the form

$$h = \begin{pmatrix} x \bar{x} & \bar{x} \\ x & 1 \end{pmatrix} = \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \otimes (x, 1) , \quad (2.71)$$

which can be characterized by having vanishing determinant,  $\det h = 0$ . Stripping away an infinite overall constant is, however, not a well defined procedure: It introduces an ambiguity of rescaling by a finite constant, which means that the coordinates  $h$  are

defined only projectively<sup>8</sup>. This is important not least because it reduces the number of degrees of freedom to the correct amount

$$\partial AdS_3 = \left\{ h \in \text{Mat}_{2 \times 2} \mid \det h = 0, \ h \sim r h \right\}. \quad (2.72)$$

Requiring homogeneity under rescaling of a boundary coordinate is a powerful constraint on amplitudes and an important clue for building the correct vertex operators. Its role is analogous to the little group scaling of spinors in the familiar spinor-helicity formalism in  $4d$ .

The boundary coordinates have a dual role as space-time coordinates and representation labels [120–124]. The boundary inherits the full action of the  $SL(2) \times SL(2)$  symmetry of left- and right-translations, which correspond to (anti-) holomorphic conformal transformations [125]. One particular basis for the right-translations, when acting on the representation of homogeneous functions in  $h$  of weight  $w$ , is given by

$$\mathcal{D}_0 = 2x \partial_x - w, \quad \mathcal{D}_+ = -x^2 \partial_x + wx, \quad \mathcal{D}_- = \partial_x, \quad (2.73)$$

for right-translations and similarly for left-translations in terms of  $\bar{x}$ . On functions of homogeneity  $w$  the quadratic Casimir  $C_2 = \mathcal{D}_0^2 + 2\mathcal{D}_+\mathcal{D}_- + 2\mathcal{D}_-\mathcal{D}_+$  evaluates to  $w(w+2)$ .

The group  $SL(2)$  carries a natural involution  $\hat{\cdot}$ , with  $\hat{\hat{g}} = g$ , which in the bulk is simply given by inversion  $\hat{g} = g^{-1}$ . On the boundary however inversion is not well defined, but there is a natural extension

$$\hat{g} = g^{-1} \text{ for } g \in SL(2), \quad \hat{h} = -\varepsilon h^T \varepsilon \text{ for } h \in \partial SL(2), \quad (2.74)$$

with the 2d Levi-Civita symbol  $\varepsilon$ . Notice that this involution exchanges left- and right-action of the group. The simplest invariant we can build this way, involving one bulk point  $g$  and one boundary point  $h$ , is

$$\varphi_\Delta(g) = \text{tr}(g \hat{h})^{-\Delta}, \quad (2.75)$$

which are known as *bulk-to-boundary propagators*. Using the relation between the connection and the structure constants (2.17) it is easily verified that this combination

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<sup>8</sup>The form (2.71) is to be understood as ‘gauge fixed’ by imposing  $h_{22} = 1$ .

satisfies the equation of motion for a scalar field

$$\left(\nabla^2 - m_\Delta^2\right) \varphi_\Delta(g) = 0 \quad (2.76)$$

with mass  $m_\Delta^2 = \Delta(\Delta-2)$ . Furthermore, as  $g$  approaches the boundary,  $\varphi_\Delta$  asymptotes to a Dirac  $\delta$ -function profile of the conformal weight  $\Delta$  centred at the point  $h$  on the boundary. These bulk-to-boundary propagators  $\varphi_\Delta$  are the natural analogue of plane waves for scattering in *AdS* [118, 126, 95, 97].

Alternative forms of the bulk to boundary propagator (2.75) are

$$\varphi_\Delta(g) = \left( \frac{e^{-\phi}}{e^{-2\phi} + (\gamma - x)(\bar{\gamma} - \bar{x})} \right)^{-\Delta} = \sum_{m, \bar{m}} V_{\Delta, m, \bar{m}}(\phi, \gamma, \bar{\gamma}) x^m \bar{x}^{\bar{m}}. \quad (2.77)$$

The ‘Fourier expansion’ in terms of  $x, \bar{x}$  has been used previously in the literature for building vertex operators, cf. [94, 127, 128] and others. We will however not find it necessary to make use of this expansion.

From the explicit expression (2.75) it is clear that the bulk-to-boundary propagator  $\varphi_\Delta$  is invariant under simultaneous left- or right-translation of  $g$  and  $h$  by a group element  $g'$  where

$$g \rightarrow gg', \quad h \rightarrow hg', \quad \hat{h} \rightarrow g^{-1} \hat{h}, \quad (2.78)$$

an likewise for left-translations. In terms of infinitesimal translation, this leads to the identity

$$j_a(z) \varphi_\Delta(g(0)) \sim -\frac{1}{z} \mathcal{D}_a \varphi_\Delta(g(0)) \quad (2.79)$$

on the bulk-to-boundary propagators. This identity is very useful for calculating correlators, as the derivative action in terms of the boundary data can be pulled out of the path integral. This relates correlators with  $j$ -insertions to the derivative with respect to the boundary coordinate of a correlator with one less  $j$ -insertion. It is worth pointing out that the order of multiplication is reversed when passing from the right-multiplication generators on the bulk point  $g$  to those on the boundary point  $h$ .

For our vertex operators we require the bulk-to-boundary propagators of supergravity states. These are of course two-tensors in the bulk, but via holography they are conserved quadratic differentials on the boundary. The graviton for instance has two on-shell degrees of freedom, corresponding to the holomorphic and anti-holomorphic stress tensor on the boundary. Using differential form notation for the bulk ( $g^{-1}dg$ ) and boundary ( $\hat{h}\partial h$  and  $\bar{\partial}h\hat{h}$ ) to encode the tensor structure, the graviton propagators



read [116, 119]

$$V_T = \varphi_4(g) \, m(g^{-1}dg, \hat{h}\partial h)^2, \quad V_{\bar{T}} = \varphi_4(g) \, m(dg \, g^{-1}, \bar{\partial}h \, \hat{h})^2. \quad (2.80)$$

Notice that these are the only non-vanishing combinations one can write down that have the correct tensor structure, homogeneity under rescaling  $h$  and are invariant under the isometries. Using (2.17), as well as  $\hat{h}\partial h \hat{h} = \hat{h}\bar{\partial}h \hat{h} = 0$ , it is readily shown that these satisfy the transversality constraints (2.66a) and supergravity equation (2.66b).

To summarize, our model contains the fixed vertex operators

$$U = c \, \tilde{c} \delta(\gamma) \delta(\tilde{\gamma}) \, \psi^a \tilde{\psi}^b \, V_{ab}(g), \quad (2.81)$$

where  $V_{ab}$  are solutions to the supergravity equations linearized around  $AdS_3 \times S^3$ . Note that the absence of a  $g \, g$  OPE means that the vertex operators built from  $V_T(g), V_{\bar{T}}(g)$  do not require normal ordering. This makes the vertex operators particularly easy to construct here, in contrast to standard string theory.

It is worth pointing out in passing that the anti-holomorphic stress tensor propagator  $V_{\bar{T}}$  is particularly simple to work with in our model. Expanding the differential  $g^{-1}dg$  in the right-invariant Cartan frame yields

$$V_T^{ab} = \varphi_4(g) \, (\hat{h}\bar{\partial}h)^a (\hat{h}\bar{\partial}h)^b, \quad (2.82)$$

so, as a function of  $g$ , the tensor structure is constant, and all the dependence on  $g$  is carried by the overall scalar propagator  $\varphi_4$ . To make contact with the form of the graviton vertex operators commonly found in the literature, notice that the natural invariant combination contained in the vertex operator reads in the basis (2.73)

$$\psi_a (\hat{h}\bar{\partial}h)^a = (x \psi_0 + \psi_+ - x^2 \psi_-) \, \bar{\partial}\bar{x} \equiv \psi(x) \, \bar{\partial}\bar{x}, \quad (2.83)$$

and the factor  $\bar{\partial}\bar{x}$  may be dropped for convenience.

Since we expect the correlators of our model to compute CFT correlators in the boundary CFT, and because the correlators of  $2d$  stress tensors are particularly simple [129], we will restrict our attention to the states (2.80) for explicit computations.

## 2.3 Correlation Functions

The chiral nature of the action means that we have a lot of control over the correlation functions in our model. We will now explain how to perform calculations in our model, in principle and with some examples, and discuss the remaining obstacle to computing arbitrary  $n$ -pt functions.

By the AdS/CFT duality we expect the worldsheet correlators of our model to compute correlators of the dual CFT [130, 131] on the boundary. At this point, we will limit ourselves to holomorphic stress tensors as external states, so the details of the boundary CFT are not important for the correlators.

Before gauging the symmetries  $G, \tilde{G}, H$  our model is completely solvable, and any correlator can be calculated exactly (non-perturbatively). By repeatedly applying the Ward identity (2.35), any (finite) number of  $j$ -insertions can be integrated out, and written as a differential operator with respect to the boundary coordinates acting on a ‘smaller’ correlator. After integrating out all  $j$ -insertions, any remaining fermions  $\psi, \tilde{\psi}$  contract into a Pfaffian each. Notice that the total number of fermion insertions is preserved under OPEs with  $j$ .

This leaves a correlator of only  $g(z)$  insertions, which then collapses onto to an integral over the zero mode  $g(z) = g_0 \in \mathcal{G}$ . This strategy provides a finite algorithm for computing any correlator in the free model and is easy to implement, even by hand.

The only non-trivial part of this algorithm is the zero mode integral. Recall the role of the  $X$  zero mode in flat space: after performing the non-zero mode integral via Wick contractions, the remaining dependence of the correlator on the zero mode is given by

$$\int d^d X_0 \prod_{i=1}^n e^{i p_i \cdot X_0} = \delta^{(d)}\left(\sum_i p_i\right), \quad (2.84)$$

yielding the momentum conserving  $\delta$ -function.

The situation is analogous in the AdS model: after performing all Wick contractions, the last part of the path integral is

$$\int_{AdS_3 \times S^3} dg_0 \prod_{i=1}^n \text{tr}(g_0 h_i)^{-\Delta_i}, \quad (2.85)$$

with the appropriate  $\Delta_i$  as fixed by the equation of motion (2.66). The  $S^3$  part of this integral can be done by standard methods, so we assume for convenience that the wavefunctions only carry dependence on  $AdS_3$ .

Just as (2.84) is the scalar  $n$ -pt contact term in flat space, the integral (2.85) is the scalar  $n$ -point contact interaction in AdS. For  $n = 2, 3$  the functional form of the answer is completely fixed by the conformal invariance, and the only unknown is a normalisation constant. Indeed, conformal invariance on the boundary implies that the answer can be written as a function of  $\frac{1}{2}n(n-3)$  independent<sup>9</sup> cross ratios. For  $n = 4$  the integral is known as the D-function, and can be expressed in terms of dilogarithms [104, 132].

There is, however, a different way of representing the integral (2.85), which has a number of benefits. It was realized by Mack [133] that the Mellin transform of CFT correlation functions with respect to the cross ratios is a significantly simpler object, being a *rational* function of the Mellin variables, times a canonical integral kernel. What's more, the so-called Mellin amplitude exhibits a recursive structure under factorisation, induced by the boundary OPE, and many more surprising and appealing features [134–139]. It also behaves very naturally in the flat space limit of taking the AdS radius to infinity, and essentially limits onto the flat space amplitude of massless particles [137].

We will make use of the Mellin space representation of the integral (2.85)

$$\int_{AdS_3} d^3 g_0 \prod_{i=1}^n \frac{1}{\text{tr}(h_i g_0)^{\Delta_i}} = \frac{\pi^h}{2} \frac{\Gamma\left(\frac{1}{2} \sum_i \Delta_i - h\right)}{\prod_i \Gamma(\Delta_i)} \int_{i\mathbb{R}+0^+} [d\delta]^{\frac{n(n-3)}{2}} \prod_{i < j} \Gamma(\delta_{ij}) h_{ij}^{-\delta_{ij}} \quad (2.86)$$

where  $h = d/2 = 1$ ,  $h_{ij} \equiv \text{tr}(h_i \hat{h}_j) = (x_i - x_j)(\bar{x}_i - \bar{x}_j)$  and the Mellin parameters  $\delta_{ij}$  are constrained to satisfy

$$\delta_{ij} = \delta_{ji} \ , \quad \sum_{j \neq i} \delta_{ij} = \Delta_i \ , \quad (2.87)$$

leaving  $\frac{1}{2}n(n-1) - n = \frac{1}{2}n(n-3)$  degrees of freedom to be integrated over. The contour of integration is to be chosen such that it runs between the semi-infinite sequences of poles in the Gamma functions, and the integral can typically be performed by a residue calculation.

The Mellin variables  $\delta_{ij}$  can be thought of as Mandelstam invariants  $\delta_{ij} = p_i \cdot p_j$  with  $p_i^2 = -\Delta_i$  in terms of some auxiliary momenta  $p_i$ . This is particularly useful in

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<sup>9</sup>In two dimensions there are  $2(n-3)$  independent cross ratios. The Mellin space representation we use, however, is valid in any number of dimensions, where there are generically  $\frac{1}{2}n(n-3)$  independent cross ratios.

the flat space limit, which can be taken essentially by setting

$$\delta_{ij} = R^2 p_i \cdot p_j, \quad p_i^2 = -\frac{1}{R^2} \Delta_i, \quad (2.88)$$

and sending the AdS radius  $R \rightarrow \infty$ . The leading piece is, up to some coupling constant dependent coefficients, the flat space scattering amplitude of massless particles with momenta  $p_i$ .

Translating more complicated correlators into Mellin space one obtains expressions of the form

$$\frac{\pi^h}{2} \frac{\Gamma\left(\frac{1}{2} \sum_i \Delta_i - h\right)}{\prod_i \Gamma(\Delta_i)} \int [d\delta]^{\frac{n(n-3)}{2}} \mathcal{M}(\delta_{ij}) \prod_{i < j} \Gamma(\delta_{ij}) h_{ij}^{-\delta_{ij}} \quad (2.89)$$

with a function  $\mathcal{M}(\delta_{ij})$  of the Mellin parameters, called the *Mellin amplitude*. The combination  $\prod_{i < j} \Gamma(\delta_{ij}) h_{ij}^{-\delta_{ij}}$  is called *Mellin kernel*, and can be thought of as analogous to the flat space momentum conserving  $\delta$ -function.

Even without taking the flat space limit, the analytical structure of the Mellin amplitude  $\mathcal{M}$  in terms of the  $\delta_{ij}$  shares many features with flat momentum space: a pure contact term is a constant, interaction terms with derivatives are polynomials and propagators lead to denominators; for an excellent review see [140]. For scalars, a set of Feynman rules for computing the Mellin amplitude has been proved [141], and progress is being made for operators with spin.

These and other observations make Mellin space a natural arena for the scattering-equations based framework which we strive to find using our ambitwistor model.

We have explained above that in the free model, any correlator with a finite number of insertions can be computed exactly via a finite algorithm. Now consider the effect of gauging the symmetries  $G, \tilde{G}, H$  and the resulting BRST gauge fixing procedure.

The gauge fixing of the fermionic symmetries is readily understood at tree level: To calculate a correlator one proceeds by inserting  $n$  of the fixed vertex operators (2.81) at locations  $z_{i=1, \dots, n}$ , as well as  $n - 2$  of each of the picture changing operators (PCOs)

$$\mathcal{Y} = \delta(\beta) G \quad \text{and} \quad \tilde{\mathcal{Y}} = \delta(\tilde{\beta}) \tilde{G} \quad (2.90)$$

at some arbitrary locations  $y_{i=1, \dots, n-2}, \tilde{y}_{i=1, \dots, n-2}$  respectively. The picture changing operators  $\mathcal{Y}, \tilde{\mathcal{Y}}$  account for the  $\beta\gamma$  and  $\tilde{\beta}\tilde{\gamma}$  moduli, respectively, and they arise in a standard way by adding the gauge fixing term  $\{Q, (\beta, \chi - \chi_0) + (\tilde{\beta}, \tilde{\chi} - \tilde{\chi}_0)\}$  to the action. The choice of locations  $y_{i=1, \dots, n-2}, \tilde{y}_{i=1, \dots, n-2}$  amounts to a choice of basis of the respective moduli spaces and drops out (only) once the gauging of the current  $H$  is fully

implemented. This is analogous to standard string theory, where the dependence on the PCO locations drops out only after integration over the moduli space, i.e. gauging of the stress tensor  $T$ .

### 2.3.1 Correlators with $n \leq 3$

For correlators with  $n \leq 3$  external states, the obstructions to gauging  $e, \tilde{e}$  to zero are actually empty, and the currents  $T, H$  are automatically<sup>10</sup> zero. Thus the final answer is given by the correlator

$$\left\langle \prod_{i=1}^n U_i(z_i) \prod_{i=1}^{n-2} \mathcal{Y}(y_i) \prod_{i=1}^{n-2} \tilde{\mathcal{Y}}(\tilde{y}_i) \right\rangle \quad (2.91)$$

for  $n = 3$ , and the same expression with an additional insertion of  $\partial c(z_1) \partial \tilde{c}(z_1)$  to saturate the zero modes for  $n = 2$ . In both cases, the dependence on the puncture locations  $z_i$  and the PCO locations  $y_i, \tilde{y}_i$  simply drops out.

For two external states  $V_{1,2}^{ab}$  our model yields the correlator

$$\langle \partial c(z_1) \partial \tilde{c}(z_1) U_1(z_1) U_2(z_2) \rangle = \int d^3g \, m_{ac} m_{bd} V_1^{ab}(g) V_2^{cd}(g) . \quad (2.92)$$

Taking the external states to be holomorphic stress tensors (2.80) this yields

$$c_2 \frac{dx_1^2 dx_2^2}{(x_1 - x_2)^4} , \quad (2.93)$$

with some normalisation constant  $c_2$ , as expected for the boundary correlator

$$\langle T(x_1) T(x_2) \rangle_{\text{boundary}} = \frac{c_{\text{bdry}}}{2} \frac{dx_1^2 dx_2^2}{(x_1 - x_2)^4} . \quad (2.94)$$

With three external states the computation is still straightforward, albeit slightly more lengthy. The naive approach to the computation is to expand the product  $U^4 G \tilde{G}$  into a sum of four terms, with 0, 1, 1, 2 insertions of  $j$ , respectively, and then apply the Ward identity (2.35) and perform Wick contractions. This is somewhat tedious, but doable. Specialising again to holomorphic stress tensors we obtain the result

$$\langle U_1(z_1) U_2(z_2) U_3(z_3) \mathcal{Y}(y) \tilde{\mathcal{Y}}(\tilde{y}) \rangle = c_3 \frac{dx_1^2 dx_2^2 dx_3^2}{(x_1 - x_2)^2 (x_2 - x_3)^2 (x_3 - x_1)^2} , \quad (2.95)$$

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<sup>10</sup>There are no quadratic differentials on the sphere with three or less simple poles.

with some normalisation constant  $c_3$ , in agreement with expected boundary correlator

$$\langle T(x_1) T(x_2) T(x_3) \rangle_{\text{boundary}} = c_{\text{bdry}} \frac{dx_1^2 dx_2^2 dx_3^2}{(x_1 - x_2)^2 (x_2 - x_3)^2 (x_3 - x_1)^2} . \quad (2.96)$$

In both cases we used the general formula (2.86) to evaluate the  $g_0$  zero mode integral. The constants  $c_{2,3}$  are not meaningful by themselves, because they can be changed by the normalisation of the operators and the overall normalisation constant of the zero-mode measure. Therefore they are not sufficient to determine the central charge of the boundary theory. We will comment further on this below.

It is worth pointing out that the 2-pt and 3-pt correlators are completely fixed by conformal symmetry, up to an overall constant. Therefore the above calculations should really be taken as a consistency check that our model is set up properly and that all the symmetries we endowed it with survive at the quantum level.

If, for instance, we had used external states that don't satisfy the transversality condition (2.66a), the dependence on the  $z_i, y_i, \tilde{y}_i$  would not have dropped out; alternatively, different ways of descending the vertex operators would have led to different coefficients  $c_{2,3}$ , signalling the inconsistency. The same would have happened if we had defined the model on a background on which the algebra  $G, \tilde{G}, H$  doesn't close at the quantum level.

### 2.3.2 Correlators with $n \geq 4$

With  $n \geq 4$  external states, the gauging of the stress tensor  $T$  and the Casimir  $H$  turns out to be a rather subtle problem. Using the gauge transformation (2.28) we can almost set to zero the corresponding gauge fields  $e, \tilde{e}$ . The obstruction to gauging them to zero arises because the fixed vertex operators require the gauge transformation to satisfy  $\alpha(z_i) = 0$ , which means  $e, \tilde{e} \in H^1(\Sigma, T(n))$ . With  $n$  fixed vertex operators this space has dimension  $n - 3$ . Going through the steps of BRST gauge fixing formally leads to the  $n$ -pt correlator

$$\int_{T^* \mathcal{M}_{0,n}} \left\langle \prod_{i=1}^n U_i(z_i) \prod_{i=1}^{n-2} \mathcal{Y}(y_i) \prod_{i=1}^{n-2} \tilde{\mathcal{Y}}(\tilde{y}_i) e^{\int e T + b de} e^{\int \tilde{e} H + \tilde{b} d\tilde{e}} \right\rangle \quad (2.97)$$

where  $(e, \tilde{e})$  are the coordinates, and the expansion of  $\int b de$  provides the measure. This expression is manifestly independent of a choice of basis in the moduli space, and is

BRST invariant in the extended sense of [51]. Picking coordinates

$$e(z) = \sum_{r=1}^{n-3} t_r e_r(z) , \quad \tilde{e}(z) = \sum_{r=1}^{n-3} a_r \tilde{e}_r(z) , \quad (2.98)$$

with  $\text{span}\{e_r\} = \text{span}\{\tilde{e}_r\} = H^1(\Sigma, T(n))$ , this turns into

$$\int_{T^*\mathcal{M}_{0,n}} \prod_{r=1}^{n-3} dt_r da_r \left\langle \prod_{i=1}^n U_i(z_i) \prod_{i=1}^{n-2} \mathcal{Y}(y_i) \tilde{\mathcal{Y}}(\tilde{y}_i) \prod_{r=1}^{n-3} e^{\int t_r e_r T + a_r \tilde{e}_r H} \int b e_r \int \tilde{b} \tilde{e}_r \right\rangle. \quad (2.99)$$

The familiar standard choice of basis for the  $e_r, \tilde{e}_r$  is  $\int e_r H = \text{Res}_r H$  for  $n-3$  of the punctures, such that the moduli  $t_r$  simply become the puncture locations<sup>11</sup>. We will choose this basis whenever convenient, but the general argument is, of course, independent of the choice of basis.

In [14] it was argued that the integral over the moduli  $(t_r, a_r)$  can be performed by appealing to Morse theory, which provides a middle dimensional cycle in the complexification of  $T^*\mathcal{M}_{0,n}$ . This is in keeping with the holomorphic nature of the model, which already requires the target space to be complexified. The integral is then shown to receive only contributions from isolated points on that cycle, which can be recognized as the solutions to the scattering equations.

A short-cut argument to arriving at the same result is to notice that the dependence on the cotangent-moduli  $a_r$  is only linearly in the exponentials, and then to declare that the integral yields  $\delta$ -functions. The rationale is that on flat Minkowski space, with plane waves as external states, the path integral over the field  $X$  can in fact be performed explicitly, which localises the conjugate field  $P$ , and therefore  $H = P^2$ , to its classical value (2.101). The exponential  $e^{\int a_r \tilde{e}_r P^2}$  can then be pulled out of the correlator, and the integral over the  $a_r$  turns into  $\delta$ -functions

$$\begin{aligned} & \int_{\mathcal{M}_{0,n}} \prod_{r=1}^{n-3} dt_r da_r e^{\int a_r \tilde{e}_r P^2} \left\langle \prod_{i=1}^n U_i(z_i) \prod_{i=1}^{n-2} \mathcal{Y}(y_i) \tilde{\mathcal{Y}}(\tilde{y}_i) \prod_{r=1}^{n-3} e^{\int t_r e_r T} \int b e_r \int \tilde{b} \tilde{e}_r \right\rangle \\ &= \int_{\mathcal{M}_{0,n}} \prod_{r=1}^{n-3} dt_r \bar{\delta}\left(\int \tilde{e}_r P^2\right) \left\langle \prod_{i=1}^n U_i(z_i) \prod_{i=1}^{n-2} \mathcal{Y}(y_i) \tilde{\mathcal{Y}}(\tilde{y}_i) \prod_{r=1}^{n-3} e^{\int t_r e_r T} \int b e_r \int \tilde{b} \tilde{e}_r \right\rangle, \end{aligned} \quad (2.100)$$

against which the remaining  $t_r$  moduli integral is to be done algebraically [13].

<sup>11</sup>From this perspective the  $z_i$  are never integrated over, but merely serve as the ‘origin’ in the integration space. The equivalence to the standard perspective is guaranteed by the fact that  $\text{Res}_i T U_i(z_i) = \partial U_i(z_i)$  and the absence of double poles.

Notice that in both lines of argument, gauging of  $T$  and  $H$  is ultimately achieved by constructing the meromorphic quadratic differential

$$P^2(w) = \sum_{i,j=1}^n \frac{p_i \cdot p_j}{(w - z_i)(w - z_j)} , \quad (2.101)$$

and solving for the locations  $z_i$  that make it vanish. The equations of motion  $p_i^2 = 0$  imply that  $P^2(w)$  only has simple poles in  $w$ , and therefore has  $n - 3$  independent components. Setting these to zero leads to  $n - 3$  equations for the  $n - 3$  moduli of the surface.

Both arguments have so far only been applied in the setting of flat space with external plane waves. The arguments in both cases don't carry over straightforwardly to our model, and any intuition gained from flat space needs to be carefully reconsidered.

Finding a consistent prescription for the moduli space integral, which is necessary to produce meaningful formulae for higher point correlators, is still an open problem. We will now discuss some observations and proposals which are currently under investigation.

In flat space the momentum field  $P_\mu(w)$  could be localised from a quantum operator to a meromorphic differential in terms of the external momenta  $p_i$ , therefore leading to the classical value of  $H = P^2$ . On  $AdS$ , the components of the field  $j_a$  do not mutually commute, so it is impossible to localise  $j_a$  to a 'number valued' meromorphic differential. It is, however, conceivable that the field  $j_a$  could localise to a differential operator acting on the external data.

Consider inserting the operator  $H$  into a correlator in the flat space mode, but with generic external states  $\varphi_i(x)$ . Using the corresponding ward identity would yield

$$\left\langle H(w) \cdots \prod_i \varphi_i(X(z_i)) \right\rangle = \int d[\phi] \left( \sum_{i,j=1}^n \frac{\partial_i \cdot \partial_j}{(w - z_i)(w - z_j)} \right) \cdots \prod_k \varphi_k(X(z_k)) \quad (2.102)$$

where  $d[\phi]$  is the CFT path integral measure. The derivatives  $\partial_i \equiv \frac{\partial}{\partial X(z_i)}$  act on the wavefunctions *before* the  $X$  path integral is performed. This can be simplified if the wavefunctions depend on some external parameter, e.g. the center of some localised wavepacket (or the boundary point in the  $AdS$  case), such that one can write

$$\frac{\partial}{\partial X^\mu} \varphi_i = -\mathcal{D}_\mu \varphi_i \quad (2.103)$$



where  $\mathcal{D}_\mu$  act on the external parameters. These can be pulled out of the path integral:

$$\left\langle H(w) \cdots \prod_i \varphi_i(X(z_i)) \right\rangle = \sum_{i,j=1}^n \frac{\mathcal{D}_i \cdot \mathcal{D}_j}{(w - z_i)(w - z_j)} \left\langle \cdots \prod_k \varphi_k(X(z_k)) \right\rangle . \quad (2.104)$$

The upshot is that, even if the operator  $j_a$  certainly cannot localise to a classical ‘number valued’ differential form, the quantum operator  $H(w)$  can under certain circumstances still localise to a classical object and thus be ‘pulled out of’ the correlator.

What’s more, the differential operator  $\mathcal{D}_i \cdot \mathcal{D}_j$  *does* have a chance to turn into a *number* when acting on appropriate states. An example of this is the Knizhnik-Zamolodchikov connection for the  $SU(2)$  WZW model, which contains an instance of the above operator, in the form  $\kappa(t^i, t^j)$ , with  $t_{\pm,0}$  the  $SU(2)$  generators. One then takes the wavefunctions to be highest weight states, which satisfy  $t_- \varphi = 0$ ,  $t_0 \varphi = \lambda \varphi$ , whereupon

$$\sum_{i,j} \frac{t_0^i t_0^j + t_+^i t_-^j + t_-^i t_+^j}{(w - z_i)(w - z_j)} \langle \cdots \rangle = \sum_{i,j} \frac{\lambda_i \lambda_j}{(w - z_i)(w - z_j)} \langle \cdots \rangle . \quad (2.105)$$

Such a setup would immediately lead to a localisation, with the analogue of the scattering equations expressed in terms of the weights  $\lambda_i$ .

Unfortunately, this particular example is too simplistic to be applicable to the  $AdS$  model. The reason is that the highest weight states of the global isometry group are not enough to generate all vertex operators since the highest weight condition is tantamount to restricting the boundary coordinate  $x_i$  to be the same for all vertex operators.

Nonetheless, if there is localisation of the moduli space integral in the  $AdS$  model, then the mechanism by which it happens will probably rely on turning the operator  $H$  into a number.

To gain further insight into this question, we consider the  $AdS$  analogue of the bi-adjoint cubic scalar with cubic interaction. Just as in flat space [31], one can set up an ambitwistor model for this by discarding the two fermions  $\psi, \tilde{\psi}$ , and instead adding two current algebras  $J_{1,2}$  for the two gauge groups. We leave the detailed discussion of this model to the future, and only cite the result for the correlator of  $n$  vertex operators

$$U = c \tilde{c} J_1^A J_2^B \frac{1}{\text{tr}(\hat{g}\hat{h})^2} , \quad (2.106)$$

which satisfy the equation of motion  $\nabla^2 = 0$  for massless scalars on  $AdS_3$ . We will treat this as a toy model for investigating localisation of the moduli integral, and hope to infer lessons that may inform the gravity model.

The vertex operators satisfy  $j_a(z)U(0) \sim -\frac{1}{z}\mathcal{D}_a U(0)$  with the differential operators (2.73) acting on the  $h_i$ , such that the correlator reads

$$\begin{aligned} & \int_{T^*\mathcal{M}_{0,n}} \prod_r dt_r da_r \left\langle \prod_r e^{t_r \int e_r T} e^{a_r \int \tilde{e}_r H} \int b e_r \int \tilde{b} \tilde{e}_r \prod_{i=1}^n U_i(z_i) \right\rangle \\ &= \int_{T^*\mathcal{M}_{0,n}} \prod_{i=4}^n dz_i da_i e^{a_i \sum_{j \neq i} \frac{\mathcal{D}^{(i)} \cdot \mathcal{D}^{(j)}}{z_i - z_j}} \left\langle c\tilde{c}(z_1) c\tilde{c}(z_2) c\tilde{c}(z_3) \prod_{i=1}^n J_1 J_2 \frac{1}{\text{tr}(g\hat{h}_i)^2}(z_i) \right\rangle \quad (2.107) \\ &= \int_{T^*\mathcal{M}_{0,n}} \frac{1}{\text{vol}SL(2)^2} \prod_{i=4}^n dz_i da_i e^{a_i \sum_{j \neq i} \frac{\mathcal{D}^{(i)} \cdot \mathcal{D}^{(j)}}{z_i - z_j}} \text{PT}_1 \text{PT}_2 \int_{AdS_3} d^3 g_0 \prod_i \frac{1}{\text{tr}(g_0 \hat{h}_i)^2} \end{aligned}$$

where from the second line we specialised to the standard basis for the  $e_r$ ,  $\tilde{e}_r$ , and  $\text{PT}_{1,2}$  are the standard world-sheet Parke-Taylor factors in terms of the  $z_i$ .

The cotangent moduli still multiply a differential operator, so the dependence of the full correlator is still not under sufficient control to perform the integral  $da_i$ . To make further progress, we proceed to evaluate the zero mode integral by going to the Mellin space representation (2.86). The correlator then becomes

$$\int_{T^*\mathcal{M}_{0,n}} \frac{1}{\text{vol}SL(2)^2} \prod_{i=4}^n dz_i da_i e^{a_i \sum_{j \neq i} \frac{\mathcal{D}^{(i)} \cdot \mathcal{D}^{(j)}}{z_i - z_j}} \text{PT}_1 \text{PT}_2 \int d\delta_{ij} \prod_{i < j} \Gamma(\delta_{ij}) h_{ij}^{-\delta_{ij}} \quad (2.108)$$

with an overall normalisation and the contour of integration as described after eq. (2.86), and  $h_{ij} = (x_i - x_j)(\bar{x}_i - \bar{x}_j)$ .

The upshot is that the differential operator  $\mathcal{D}^{(i)} \cdot \mathcal{D}^{(j)}$  actually turns into a *number* when acting on the Mellin kernel! Indeed, using the representation (2.73) we obtain

$$\mathcal{D}^{(i)} \cdot \mathcal{D}^{(j)} = \delta_{ij} + 2 - \sum_{\substack{m \neq i \\ n \neq j}} \delta_{im} \delta_{jn} \frac{(x_i - x_j)(x_m - x_n)}{(x_i - x_m)(x_j - x_n)} =: \mathcal{S}_{ij}. \quad (2.109)$$

Notice that these coefficients need to satisfy  $\sum_{j \neq i} \mathcal{S}_{ij} = 0$  in order for  $j^2$  to be free of double poles, and this is guaranteed by  $\sum_{j \neq i} \delta_{ij} = 2$ .

All this means that in this toy model the dependence of the correlator on the cotangent moduli is the same as in flat space, and one could expect the correlator to

localise *inside* the Mellin integral

$$\int d\delta_{ij} \prod_{i<j} \Gamma(\delta_{ij}) h_{ij}^{-\delta_{ij}} \int_{\mathcal{M}_{0,n}} \frac{1}{\text{vol}SL(2)^2} \prod_{i=4}^n dz_i \bar{\delta} \left( \sum_{j \neq i} \frac{\mathcal{S}_{ij}}{z_i - z_j} \right) \text{PT}_1 \text{PT}_2 . \quad (2.110)$$

The worldsheet integral can be performed at arbitrary  $n$  and turns into the sum over cubic Feynman graphs with constant vertices and propagators given by sums of  $\mathcal{S}_{ij}$  in the denominator [142, 42].

While the result (2.110) is interesting in its own right, we defer a detailed study to the future. Here it only serves as a toy model for how the moduli space integral in  $AdS$  might end up localising, and what the potential scattering equations might look like.

The study of the  $n$ -point correlators in the  $AdS$  gravity model is still ongoing, and it is not clear if or how a localisation to  $AdS$  scattering equation happens. Explicit computations of the first few terms of the expansion of the exponential  $e^{\int \bar{\epsilon} H}$  in the 4-point correlator suggest that the dependence on the cotangent moduli  $a_r$  is not a simple exponential, as in the flat space model or the scalar toy model. We do however observe drastic simplifications when going to Mellin space, and it may be possible to make further progress by using the generalised Mellin amplitude formalism of [143], which naturally arises in the study of correlators involving operators with spin.

## 2.4 Discussion

We have initiated the study of the ambitwistor string on a group manifold. We examined the classical theory on an arbitrary group and discussed the consistency conditions on the background that arise via from quantisation. Choosing the particular background  $AdS_3 \times S^3$  with NS flux, we found that the BRST cohomology is made up of the solutions to the supergravity equations of motion, linearised around this background.

The chiral nature of the action is familiar from ambitwistor models, but strange from the perspective of standard string theory. Just as in the standard WZW type models, there is a strong link between chirality on the worldsheet and left/right translations in the target. Recall that in a WZW model [144, 145, 101] the Kac-Moody generators for left/right translations given in terms of the fundamental field  $g(z, \bar{z})$  as

$$j_R = g^{-1} \partial g , \quad j_L = (\bar{\partial} g) g^{-1} , \quad (2.111)$$

with the properties  $\bar{\partial}j_R = \partial j_L = 0$ , up to contact terms, as a consequence of the equations of motion. In the present ambitwistor model, the left/right generators are given by

$$j_R = j, \quad j_L = g(j + \dots)g^{-1} \quad (2.112)$$

where the dots are corrections necessary to make the Kac-Moody algebras consistent.

The lack of a fundamental field generating left-translations is an echo of the chiral nature of this ambitwistor string.

Classically one can exchange the roles of left/right translations by a change of variables involving  $g \rightarrow g^{-1}$ , but at the quantum level, this can lead to subtle effects. The gravity model we defined is left/right symmetric even at the quantum level, but the bosonic toy model mentioned briefly is not.

We have computed the 2- and 3-pt functions for holomorphic stress tensors in our model. Computation of a higher point amplitude will start requiring an *AdS* analogue of the scattering equations. We have illustrated possible avenues and obstacles to a scattering equations based framework for *AdS* correlators and illustrated this with a scalar toy model.

We found striking evidence that the Mellin representation space may be crucial to understanding ambitwistors in *AdS*. Already in flat space, localisation in the moduli space only happens in the special basis of plane waves for external states. The closest analogue to this in *AdS* is indeed Mellin space, which essentially limits to the space of Mandelstam invariants in the flat space limit.

The scalar toy model localises on a new type of scattering equations when expressed in *Mellin space*, which is not surprising given the close kinship to flat space momentum space. It is the form of the scattering equations that is however rather surprising. While we defer a detailed discussion of this model and its scattering equations to the future, we want to point out one salient feature of the solutions to these scattering equations. They have a very natural behaviour under the boundary OPE: whenever two boundary points come close,  $x_{ij} \rightarrow 0$ , the scattering equations force the corresponding punctures to collide,  $z_{ij} \rightarrow 0$ , linearly.

This is reminiscent of the flat space scattering equations, where going to a factorisation channel by tuning the Mandelstam invariants forces the corresponding vertex operators to approach. This establishes a link between the kinematical factorisation of the target space process and the geometric factorisation of the worldsheet.

It is pleasant to see that this link continues to hold in the *AdS* scattering equations.

It is worth mentioning that one can readily define the full zoo of ambitwistor theories [31] on the  $AdS_3 \times S^3$  background. Of these, the heterotic model is particularly interesting, because it is still entirely analytically tractable, and all  $n$ -pt correlators can be computed in closed form. In many ways, it interpolates between the simplistic scalar model and the fully fledged gravity model.

The correlator of  $n$  holomorphic currents in the heterotic toy model is given by the replacing one of the Parke-Taylor factors in the bi-adjoint scalar correlator (2.108) by

$$\text{PT}(\{z_i\}) \rightarrow \prod_{i=3}^n \left( \sum_{j \neq i} \frac{\varepsilon_i \cdot \mathcal{D}^{(j)}}{z_i - z_j} \right) \frac{m(\varepsilon_1, \varepsilon_2)}{z_1 - z_2}$$

with the polarization vectors  $\varepsilon_i^a = \langle x_i | t^a | x_i \rangle$ , and the derivative operators  $\mathcal{D}$  acting on the Mellin kernel.<sup>12</sup> The order of the product does not matter. In the flat space limit, this expression formally reassembles into the familiar CHY Pfaffian, but the non-commutativity of the generators  $\mathcal{D}$  means a more sophisticated interpretation may be needed on  $AdS$ .

The dependence on the cotangent moduli of this correlator is not as simple as in the scalar toy model. While going to Mellin space does lead to a significant simplification, turning the derivative-valued operator  $H$  into an algebraic one it is not clear if or how this eventually leads to a localisation of the moduli space integral.

We will discuss this model in detail in an upcoming publication.

An essential computation of interest which has yet to be done in the gravity model is the computation of the boundary central charge. The correlators of holomorphic stress tensors are expected to only compute the connected part of the boundary correlators, so they are not sufficient to determine the central charge unambiguously. The same happens in standard string theory [116]. The approach taken e.g. in [94] proceeds by constructing the boundary Virasoro generators directly in terms of conserved currents on the worldsheet. The connection between the two methods is that the graviton vertex operator can formally be written as a BRST variation, but the integration by parts on the moduli space yields the Virasoro generator as boundary term [116]. This is the worldsheet echo of the well-known subtlety that lies at the core of the Brown-Henneaux construction. It would be interesting to understand the analogous argument in our model.

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<sup>12</sup>It is tempting to conjecture that the gravity amplitude is simply given by replacing both Parke-Taylor factors with two copies of this structure. This is exactly what happens in flat space, and a manifestation of the “double copy” structure there, but appears to be too naive and leads to several inconsistencies.

More generally, for correlators involving not only stress tensors, the details of the dual CFT [130, 131] should start to emerge, and a better understanding from the ambitwistor worldsheet would be desirable.

In the same vein, it would be fascinating to understand better the emergence of the boundary OPE from the ambitwistor worldsheet, in particular with regards to any potential scattering equations. The study of the scalar and heterotic toy model have already started to shed light on this.

Understanding the emergence of the boundary OPE is also likely going to be the way towards proving the  $n$ -point correlator expression (2.99). The natural approach to this would seek to establish a recursive structure in the Mellin amplitudes, which is known to be a manifestation of the boundary OPE [137].

A very exciting, yet vague, prospect arises from the fact that the Mellin space representation 2.86 is actually valid in any number of dimensions. If our model can be used to derive a scattering equations based formula for gravity, it could potentially be generalized to higher dimensions, despite being derived originally in  $AdS_3$ .

This hope is inspired by flat space, where the model requires the target space dimension  $d = 10$ , but after computing a (tree level) correlator, the final result is valid in any dimension. A more subtle statement appears to be true even for loop amplitudes [18–20].

Clues as to whether this is possible could come from making contact with the twistor string formula for scattering on  $AdS_4$  in [146].

## Chapter 3

# Ambitwistor Strings at Loop Level – The Glueing Operator

In this section, we present a new operator in the ambitwistor string which glues together correlators with fewer points or of lower genus. It underpins the recursive construction of tree-level CHY scattering amplitudes by Dolan & Goddard, as well as the computation of loop integrands on a Riemann sphere by Geyer et al.

The gluing operator turns out to be a tractable object due to the finiteness of the spectrum: The construction of higher point amplitudes by sewing together two Riemann surfaces, or higher loop amplitudes by self-sewing a Riemann surface, is well established in the operator approach to standard string theory (see, e.g., [147, 148]). Non-chiral strings contain an infinite number of states in their BRST spectrum, so the standard string propagator is rather difficult to handle. Since the ambitwistor string spectrum is just that of massless field theory, its propagator should be correspondingly simpler.

One of the most immediate advantages of the existence of the ambitwistor string theory [13] which gives rise to the CHY formula for gravity is that it gives a clear recipe for computing higher loop amplitudes by putting the theory on a higher genus surface [26, 149]. Around flat space-time, correlation functions of the worldsheet CFT again localise on solutions to higher genus scattering equations [26, 18] so that the integral over the moduli space of higher genus curves again amounts to summing the correlator over solutions to these equations, which now fix the worldsheet complex structure in addition to the location of the vertex operators. After summing over all solutions to these equations (and, in the case of RNS-type models, summing over worldsheet spin structures) the integrand is again a rational function of the external data – as expected for the integrand of a field theory. This rational function has

been explicitly computed at four points, where for  $g = 1, 2$  it coincides with the 1-loop [18, 19] and 2-loop [149] supergravity integrands, respectively. At higher points it has been shown that the  $g = 1$  ambitwistor string has the correct factorisation properties [26, 149] and correct behaviour as one of the gravitons becomes soft [150], as expected for supergravity. The loop integrals themselves arise from the zero-modes of worldsheet field  $P_\mu(z)$ . It is worth pointing out that since the moduli space of the ambitwistor string is non-compact, unlike that of the standard string, there is a natural place for UV divergences to arise.

Since the ambitwistor correlation functions are computed on a curve of genus  $g > 0$  *before* summing over solutions to the scattering equations, they are naturally written in terms of Riemann theta functions. One does not expect to find such theta functions appearing in a *field theory* amplitude, suggesting that there should be a simpler way of rewriting the ambitwistor correlation function. This was found by Geyer *et al.* in [18, 19], following earlier work of Dolan & Goddard [87] related to BCFW recursion of the tree-level CHY formulae. In [18, 19] it was shown that, instead of localising the ambitwistor string to solutions of the higher genus scattering equations, one could localise to the boundary of the  $g = 1$  moduli space corresponding to a non-separating degeneration. This is achieved using the global residue theorem in  $\overline{\mathcal{M}}_{1,n}$  to take the contour to surround this boundary divisor – where the ambitwistor string integrand again has a simple pole – instead of one of the poles of the scattering equations. The advantage of this approach is that, on this non-separating boundary divisor, the integrand can be expressed in terms of functions on the nodal Riemann sphere, much more closely in line with what one expects from a field theory Feynman graph.

It was shown that the one-loop result can be derived from a dimensional reduction of an amplitude in the forward limit [15, 151].

We propose that the loop integrands can also be computed directly from a correlation function in the  $g = 0$  ambitwistor string on nodal Riemann surfaces. As well as vertex operators representing the external states, the correlation function also involves a new operator  $\Delta(z, w)$  that we call the *gluing operator*. This gluing operator plays the role of the propagator in the target space field theory. It is surprising that an inherently off-shell object such as a field theory propagator can be represented by a BRST invariant insertion. Indeed, as discussed above, local operators in the BRST cohomology of the ambitwistor string represent *on-shell* states of 10d SUGRA, so since the gluing operator represents an off-shell propagator, it cannot be an element of the BRST cohomology of local operators. It seems natural to give up the condition of



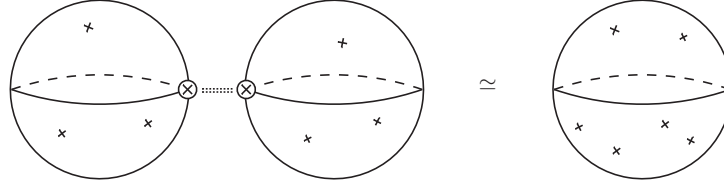


Fig. 3.1 Gluing two genus zero correlators, forming a new sphere.

locality (rather than BRST invariance), and indeed the gluing operator is non-local, while it retains full BRST invariance.

Although  $\Delta(z, w)$  is genuinely non-local on  $\Sigma$  (rather than just bi-local), the two points  $z$  and  $w$  play a special role. These two points are each associated with the insertion of a set of local operators that now correspond to ‘off-shell states’. The role of these operators can be understood as follows. The relation to the target space propagator dictates the role of the gluing operator in a factorisation limit of the original amplitude. More precisely, unitarity demands that, whenever the corresponding target space propagator goes on-shell, the gluing operator has a simple pole with residue given by the insertion of a complete set of states in the Hilbert space. By the state-operator correspondence, this can be implemented by a sum over a complete set of local vertex operators in the BRST cohomology, representing the on-shell particle flowing out of one node and into the other. Away from the factorisation channel, we must extend the local operators off-shell, and by themselves, they cannot be BRST invariant. The failure of these off-shell insertions to be BRST invariant is fully compensated by the remaining, non-local pieces of  $\Delta(z, w)$ .

These two special points may be inserted on different curve components, corresponding to divisors in the boundary of the moduli space describing separating degenerations, or both on the same curve component for a non-separating degeneration. The insertion of  $\Delta(z, w)$  operator can thus intuitively be thought of as identifying  $z$  and  $w$ , thus changing the topology of the worldsheet, as well as transporting the CFT data from one node to the other.

In particular, we claim that the full,  $n$ -particle tree amplitude can be computed by using  $\Delta(z, z')$  to glue together two sub-amplitudes, each with one leg off-shell as

$$\int_{\overline{\mathcal{M}}_{0,n}} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle_{\Sigma} = \sum_{\substack{\text{channels} \\ L \cup R = \{1, \dots, n\}}} \int_{\partial \overline{\mathcal{M}}_{0,n}} \left\langle \mathcal{O}^{(s)}(z_*) \prod_{i \in L} \mathcal{O}_i(z_i) \right\rangle_{\Sigma_L} \left\langle \mathcal{O}_{(s)}(z'_*) \prod_{j \in R} \mathcal{O}_j(z'_j) \right\rangle_{\Sigma_R}, \quad (3.1)$$

where  $\Delta(z, z') \sim \mathcal{O}^{(s)}(z) \mathcal{O}_{(s)}(z')$  is a schematic representation of the gluing operator, whose detailed form will be given below. This operator is inserted at the nodes (with

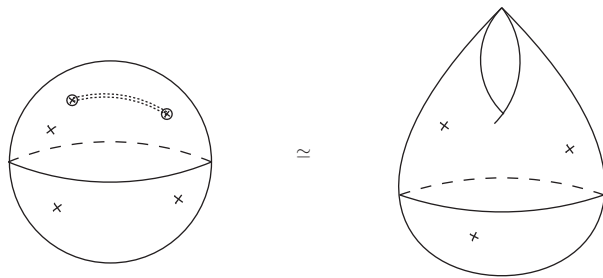


Fig. 3.2 The gluing operator identifies two points in a genus zero correlator, forming a nodal sphere.

coordinates  $z_*$  and  $z'_*$  on  $\Sigma_{L/R}$ , respectively) and here has one ‘leg’ on the left curve component and one on the right, as in fig. 3.1. The correlation functions on the *rhs* each correspond to tree amplitudes extended to allow the leg associated to  $z_*/z'_*$  to go off-shell<sup>1</sup>. The sum over channels in (3.1) will be explained in more detail below, but it is essentially a sum over all the different boundary divisors in  $\overline{\mathcal{M}}_{0,n}$  where the original integrand had a single pole.

Similarly, we claim that the 1-loop integrands of [18] are in fact correlators of the  $g = 0$  ambitwistor string CFT computed by inserting the gluing operator with both ‘legs’ on the same sphere, in addition to the usual vertex operators eq. (1.36). Schematically, we can express this as

$$\int_{\overline{\mathcal{M}}_{1,n}} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle_{\Sigma} = \int_{\overline{\mathcal{M}}_{0,n+2}} \langle \Delta(z_+, z_-) \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle_{\Sigma} \quad (3.2)$$

where  $z_{\pm}$  the locations of the nodes. (See fig. 3.2.)

The plan of this section is as follows. Section 3.1 gives a detailed description of the gluing operator in bi-adjoint  $\phi^3$  theory at tree-level, showing how it can be used to reconstruct tree amplitudes in CHY form. We move to demonstrate how its insertion into the  $g = 0$  ambitwistor string path integral generates the nodal sphere form of the loop integrand found by [18, 19], for cubic scalar, Yang-Mills and gravity in sections 3.1, 3.2 and 3.4 respectively.

We stress that, unlike the Type II gravity ambitwistor string, the (known) models containing such current algebras are not consistent. Nonetheless, we will find it convenient to consider such anomalous models below, so as to illustrate the gluing operator in a simpler context.

<sup>1</sup>Off-shell continuations of CHY amplitudes have previously been considered e.g. in [87, 152, 153].

### 3.1 The gluing operator for the bi-adjoint scalar

We now begin our presentation of the detailed form of the gluing operator  $\Delta(z_*, z'_*)$ . In this section we will consider the gluing operator for the bi-adjoint scalar theory, demonstrating its use in computing tree and 1-loop amplitudes, each in a given colour ordering. As explained in the introduction, for tree amplitudes,  $z_*$  and  $z'_*$  should be chosen to be points on separate Riemann spheres, joined by this operator, while for the 1-loop amplitude they will both be inserted on the same Riemann sphere.

The gluing operator for the bi-adjoint scalar theory takes the form

$$\Delta_{\phi^3}(z_*, z'_*) = \int \frac{d^d \ell}{\ell^2} W_{ij}(z_*) W_{kl}(z'_*) \mathcal{O}^{a\bar{a}}(z_*) \Delta_{ab\bar{a}\bar{b}} \mathcal{O}^{b\bar{b}}(z'_*) , \quad (3.3)$$

where  $i, j, k, l \in \{1, \dots, n\}$  are four external punctures with a special role, described below. The operators  $\mathcal{O}^{a\bar{a}}(z_*)$  and  $\mathcal{O}^{b\bar{b}}(z'_*)$  are local insertions on the left/right sphere, respectively, defined by

$$\begin{aligned} \mathcal{O}^{a\bar{a}}(z_*) &= c \tilde{c} j^a \bar{j}^{\bar{a}} e^{i\ell \cdot X}(z_*) \\ \mathcal{O}^{b\bar{b}}(z'_*) &= c \tilde{c} j^b \bar{j}^{\bar{b}} e^{-i\ell \cdot X}(z'_*) . \end{aligned} \quad (3.4)$$

Note the signs of the momenta in the exponentials, corresponding to off-shell momentum  $\ell$  flowing into the left Riemann sphere and out of the right. The only differences between  $\mathcal{O}^{a\bar{a}}$  and the local on-shell vertex operators (1.43) are that the momentum  $\ell$  in (3.4) is not required to be null, and that the operators in (3.4) have arbitrary colour. The operators are joined by the tensor structure of the target space Feynman propagator for a scalar

$$\Delta_{ab\bar{a}\bar{b}} = \delta_{ab} \delta_{\bar{a}\bar{b}} \quad (3.5)$$

which connects the colour flow from one Riemann sphere to the other<sup>2</sup>. Thus, this part of the gluing operator represents a set of ‘off-shell states’ flowing through the node.

Since these ingredients describe off-shell states they cannot, by themselves, be invariant under the original BRST operator on each Riemann sphere separately. The failure of  $\mathcal{O}^{a\bar{a}}(z_*)$  to be BRST closed is compensated on each side separately by the remaining ingredient

$$W_{ij}(z_*) = \exp \left( \frac{\ell^2}{2} \int_{\Sigma} e(x) \omega_{i*}(x) \omega_{j*}(x) \right) \quad (3.6)$$

<sup>2</sup>In (3.5) we have given the colour structure for a  $U(N)$  gauge group. However, since the colour singlet decouples by the Kleiss-Kuijff relations, this also holds for  $SU(N)$ .

which is a non-local operator.  $W_{ij}(z_*)$  depends on  $z_*$  (as well as  $z_i, z_j$ ) through the meromorphic 1-forms  $\omega_{i*}$  and  $\omega_{j*}$ . Crucially,  $W_{ij}(z_*)$  carries dependence on the gauge field  $e$ , and hence will modify effective BRST operator to

$$Q_{\text{eff}} = \oint c T + \frac{\tilde{c}}{2} (P^2 - \ell^2 \omega_{i*} \omega_{j*}) \quad (3.7)$$

on the left Riemann sphere, with a similar effective BRST operator on  $\Sigma_R$ . We emphasise that the fundamental BRST operator (1.34) remains unchanged; this modified *effective* BRST operator emerges naturally after integrating out the gauge fields and applying standard BRST quantisation in the presence of the glueing operator.

### 3.1.1 Tree amplitudes in $\phi^3$ theory

Using this glueing operator, we claim that at tree-level the colour-ordered partial amplitudes  $\mathbf{m}(\alpha, \beta)$  in the cubic bi-adjoint scalar theory can be written as

$$\mathbf{m}(\alpha, \beta) = \sum_{\substack{\alpha_L \cup \alpha_R = \alpha \\ \beta_L \cup \beta_R = \beta \\ \text{compatible}}} \bar{\mathbf{m}}^{a\tilde{a}}(\alpha_L, \beta_L) \Delta_{ab\tilde{a}\tilde{b}} \bar{\mathbf{m}}^{b\tilde{b}}(\alpha_R, \beta_R) \quad (3.8)$$

where  $\bar{\mathbf{m}}^{a\tilde{a}}(\alpha_L, \beta_L)$  denotes the correlator

$$\bar{\mathbf{m}}^{a\tilde{a}}(\alpha_L, \beta_L) = \int_{\mathcal{M}_{0, n_L+1}} \left\langle W_{ij}(z_*) \mathcal{O}^{a\tilde{a}}(z_*) \prod_{i \in L} \mathcal{O}_i(z_i) \right\rangle \Big|_{\alpha_L, \beta_L}, \quad (3.9)$$

involving insertions of one leg of the glueing operator, together with the vertex operators for the particles that are on the ‘left’ Riemann sphere (similarly for  $\bar{\mathbf{m}}^{b\tilde{b}}(\alpha_R, \beta_R)$ ). The original  $n$ -particle colour ordering  $\alpha$  has split into two  $\alpha_{L/R} \in S_{n_{L/R}+1}$ , where the extra element is the node, and likewise for  $\beta$ . (The subscripts on the correlator instruct us to take just these colour orderings.) The split of particles into left and right has to be *compatible* with the original colour ordering, in the sense that stepping through the entire set, from  $\alpha(i)$  to  $\alpha(i+1)$ , must require crossing from the left to the right sphere, and back, exactly once. The sum over channels means a sum over all ways of splitting the external particles into two ordered sets, compatible with colour ordering in the way just described, and with the property that exactly two out of the four particles  $i, j, k, l$  are on each side. We have to make a choice of  $i, j, k, l$ , but the sum is independent of this choice - as we explain below.

The correlator (3.9) represents a partial amplitude with one leg off-shell. It can be easily evaluated. Integrating out  $X$  from the Riemann sphere on the left, we find as usual a momentum-conserving  $\delta$ -function that sets

$$\ell + \sum_{i \in L} p_i = 0 , \quad (3.10)$$

and that

$$P_L(z) = \sum_{i \in L} p_i \omega_{i*}(z) . \quad (3.11)$$

Here, we have used (3.10) to eliminate the explicit dependence of  $\ell$  from  $P_L(z)$ ; note that it is now important that each term in the sum has a simple pole at the node  $z_*$ .

The presence of the gauge field  $e$  in the gluing operator alters the scattering equations. Integrating it out, we now find that  $\bar{\mathbf{m}}$  on the left is supported on solutions of the ‘off-shell scattering equations’

$$P_L^2(z) = \ell^2 \omega_{i*}(z) \omega_{j*}(z) \quad (3.12)$$

Similarly, we obtain from the right Riemann sphere a momentum conserving  $\delta$ -function fixing  $-\ell + \sum_{i \in R} p_i = 0$ , which together with (3.10) implies overall momentum conservation, as well as the constraints

$$P_R^2(z') = \ell^2 \omega_{k*}(z') \omega_{l*}(z') \quad \text{with} \quad P_R(z') = \sum_{i \in R} p_i \omega_{i*}(z') \quad (3.13)$$

on the right hand sphere. Note that these conditions, together with the definition of  $P_L(z)$ , are exactly what we would obtain in the limit that the original Riemann sphere degenerates to form a node.

After performing the CFT path integral to compute the correlator, we find explicitly

$$\bar{\mathbf{m}}^{a\tilde{a}}(\alpha_L, \beta_L) = \int \left( \frac{1}{\omega_{ij*}} \right)^2 \prod_{i \in L \setminus \{i,j\}} \bar{\delta}(\text{Res}_{z_i} P_L^2) \text{PT}(\alpha_L)^a \text{PT}(\beta_L)^{\tilde{a}} \quad (3.14)$$

for the left building block, and similarly for the right. The Parke-Taylor factors with an index are defined as

$$\text{PT}(\alpha)^a = \text{tr} \left( t_{\alpha(1)} \cdots t_{\alpha(n)} t^a \right) \prod_{i=1}^{n-1} S(z_{\alpha(i)}, z_{\alpha(i+1)}) S(z_{\alpha(n)}, z_*) S(z_*, z_{\alpha(1)}) , \quad (3.15)$$

which is the straightforward generalisation of eq. (1.42), with the node inserted as a new member of the colour trace. As required by unitarity, when the momentum flowing through the node goes on-shell, the  $\bar{\mathbf{m}}$  become proper on-shell amplitudes, as the gluing operator simply turns into a pair of on-shell vertex-operators traced over the on-shell Hilbert space.

Let us now explain why, with these ingredients, the sum (3.8) indeed agrees with the original  $n$ -partial colour ordered amplitude  $\mathbf{m}(\alpha, \beta)$  in (1.45). We begin by integrating (1.45) by parts in the moduli space  $\overline{\mathcal{M}}_{0,n}$ , and take for concreteness  $i, j, k, l = 1, 2, 3, 4$ . Explicitly, we write

$$\begin{aligned} \mathbf{m}(\alpha, \beta) &= \int \left( \frac{1}{\omega_{123}} \right)^2 \bar{\partial} \frac{1}{\text{Res}_{z_4} P^2} \prod_{i=5}^n \bar{\delta}(\text{Res}_{z_i} P^2) \text{PT}(\alpha) \text{PT}(\beta) \\ &= - \int \frac{1}{\text{Res}_{z_4} P^2} \bar{\partial} \left[ \left( \frac{1}{\omega_{123}} \right)^2 \prod_{i=5}^n \bar{\delta}(\text{Res}_{z_i} P^2) \text{PT}(\alpha) \text{PT}(\beta) \right] \end{aligned} \quad (3.16)$$

so that the  $\bar{\partial}$  derivative originally acting on  $\text{Res}_{z_4} P^2$  now picks up contributions from any poles in the rest of the integrand. (Equivalently, treating the original integrand as a top meromorphic form on  $\overline{\mathcal{M}}_{0,n}$ , we deform the contour, originally surrounding all scattering equations, away from the pole at  $\text{Res}_{z_4} P^2 = 0$  and thus pick up residues from all other poles outside the original contour.) As shown in [87], the only poles of the expression in square brackets on (3.16) lie on the boundary of the moduli space  $\overline{\mathcal{M}}_{0,n}$  where the Riemann sphere degenerates to a nodal curve with two components. (Recall that we continue to impose the scattering equations for particles  $5, \dots, n$ .) Furthermore, we obtain at most a simple pole on boundary divisors that are compatible with the colour orderings  $(\alpha, \beta)$  in the sense given above, and in which exactly two of the distinguished points  $z_1, z_2, z_3, z_4$  lie on each curve component.

It is worth emphasising that the ambitwistor correlation functions are independent of the choice of  $n - 3$  points at which we choose to impose scattering equations only as long as *all* the scattering equations are enforced. In order to employ the global residue theorem, we have to *choose* a meromorphic form that extends the original CHY integrand off the support of the scattering equations; the form given in the second line of (3.16) certainly achieves this, but it is far from the unique choice. Each such expression has the same residue at the solutions to the scattering equations, but their value/residues may differ significantly everywhere else. Furthermore, as soon as we relax one of the scattering equations, (3.16) ceases to be independent of the choice of which points have, or do not have, scattering equations imposed. Thus, as soon as we

integrate by parts away from the pole at  $\text{Res}_{z_4} P^2$  in eq. (3.16), we lose invariance under the permutations of the external particles; we have singled out the particles at  $z_{1,2,3,4}$  as playing a special role.

More generally, we could talk of independence of the choice of basis of

$$T^{1,0} \mathcal{M}_{0,n} \cong H^{0,1}(\Sigma, T_\Sigma(-z_1 - \cdots - z_n))$$

that we use to describe the  $\tilde{b}$ -ghost moduli responsible for imposing the scattering equations. Fundamentally, this is because the independence of the points  $w_r$  is a reflection of the gauge invariance under local transformation generated the charge  $P^2$ ; by going away from the locus of the solutions to the scattering equations, we are breaking this gauge invariance.

Coming back to our example, after relaxing the scattering equation at  $z_4$ , the field  $P(z)$  no longer obeys  $P^2(z) = 0$  for all  $z \in \Sigma$ , since  $n - 4$  scattering equations are not enough to enforce  $P^2 = 0$  globally. The remaining scattering equations

$$\text{Res}_{z_i} P^2 = 0, \quad \text{for } i = 5, \dots, n, \quad (3.17)$$

are however enough to imply that

$$P^2(z) = \frac{(w - z_1)(w - z_2)(w - z_3)(w - z_4)}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)} \frac{(dz)^2}{(dw)^2} P^2(w) \quad (3.18)$$

where  $w$  is a fixed, arbitrary point anywhere on the sphere, and the RHS is independent<sup>3</sup> of  $w$  by Liouville's theorem: it is a scalar function of  $w$  with no poles. Recall that the equations (3.17) are to be understood as a constraint on the moduli of the surface. They can be thought of as fixing all but one of the locations  $z_i$ , up to Möbius invariance.

Let us consider the behaviour of  $P^2(z)$  as we approach a degeneration in which points 1, 2 lie on the 'left' component curve while 3, 4 lie on the 'right'. In this limit, eq. (3.18) becomes

$$P_L^2(z) = \frac{(w - z_1)(w - z_2)(w - z_*)(w - z_*)}{(z - z_1)(z - z_2)(z - z_*)(z - z_*)} \frac{(dz)^2}{(dw)^2} P_L^2(w), \quad \text{for } w, z \in \Sigma_L, \quad (3.20)$$

<sup>3</sup>In other words, on the support of the scattering equations  $\text{Res}_{z_i} P^2 = 0$  for  $i = 5, \dots, n$ , the combination

$$P^2(z) \frac{1}{dz^2} (z - z_1)(z - z_2)(z - z_3)(z - z_4) \quad (3.19)$$

is independent of  $z$ , (though still depends on the  $z_i$ ), which again follows from Liouville's theorem.

and similarly for  $w, z \in \Sigma_R$ . Recall that  $z_*$  denotes the location of the emergent node. Note that in going from (3.18) to (3.20), the RHS remains independent of the arbitrary point  $w$  throughout. We can simplify (3.20) by taking the limit  $w \rightarrow z_*$ , where it becomes

$$P_L^2(z) = \ell^2 \omega_{1*}(z) \omega_{2*}(z) , \quad (3.21)$$

with  $\ell = -\sum_{i \in L} p_i = \sum_{i \in R} p_i$  the momentum flowing through the node. Similarly, on the ‘right’ curve component, with local coordinate  $z'$ , we have

$$P_R^2(z') = \ell^2 \omega_{3*}(z') \omega_{4*}(z') . \quad (3.22)$$

These are exactly the off shell scattering equations (3.12) that arose above from integrating out the gauge field  $e$  in the presence of the gluing operator. Furthermore, from (3.22) it follows that on this boundary divisor

$$\frac{1}{\text{Res}_{z_4} P^2} = \frac{1}{\ell^2 \omega_{3*}(4)} . \quad (3.23)$$

We recognise the factor of  $1/\ell^2$  as present in the gluing operator, while the factor of  $\omega_{3*}(4)$  combines with other ingredients to form the volume of the Möbius group on the right Riemann sphere. Finally, the limit of each worldsheet Parke–Taylor factors becomes the coloured Parke–Taylor factor (3.15) describing the colour flow through the node. We thus see that the sum over all compatible boundary divisors

$$\sum_{\substack{\alpha_L \cup \alpha_R = \alpha \\ \beta_L \cup \beta_R = \beta \\ \text{compatible}}} \bar{\mathbf{m}}^{a\bar{a}}(\alpha_L, \beta_L) \Delta_{ab\bar{a}\bar{b}} \bar{\mathbf{m}}^{b\bar{b}}(\alpha_R, \beta_R)$$

given in (3.8) indeed corresponds to an evaluation of the original colour–ordered amplitude  $\mathbf{m}(\alpha, \beta)$ . We emphasise that this sum gives the *complete* (colour–ordered) amplitude, not its cuts. Of course, by varying the external momenta and taking the residue as  $\ell^2 \rightarrow 0$  we indeed would obtain the cut amplitude in a given channel. We have checked numerically up to six points that this construction indeed coincides with the original amplitude.

### 3.1.2 One loop amplitudes in $\phi^3$ theory

To obtain 1-loop amplitudes in this bi-adjoint theory we use essentially the same gluing operator as at tree level, but now with both ‘legs’ inserted on the same copy of a Riemann sphere. Specifically, along with the  $n$  vertex operators describing external



states, we insert

$$\Delta_{\phi^3}(z_+, z_-) = \int \frac{d^d \ell}{\ell^2} \mathcal{O}^{a\tilde{a}}(z_+) \Delta_{ab\tilde{a}\tilde{b}} \mathcal{O}^{b\tilde{b}}(z_-) W(z_+, z_-) \quad (3.24)$$

where  $\mathcal{O}^{a\tilde{a}}$  and  $\Delta_{ab\tilde{a}\tilde{b}}$  were defined in (3.4) and (3.5) above, while now

$$W(z_+, z_-) = \exp \left( \frac{\ell^2}{2} \int_{\Sigma} e(x) \omega_{+-}^2(x) \right) . \quad (3.25)$$

in place of (3.6). Note that in both the tree and 1-loop factors, the  $W$ s can be understood as describing moduli associated to the normal bundle of the boundary divisor. It can also be seen as a holomorphic Wilson line [154–156], transporting a frame for the gauge field  $e$  from  $z_+$  to  $z_-$ .

Let us first understand the emergence of the one-loop scattering equations. This follows in much the same way as the modification of the scattering equations at tree level. In the presence of the gluing operator, the  $PX$  path integral reads

$$\int \mathcal{D}P \mathcal{D}X e^{-\int P \bar{\partial} X + \frac{\ell}{2} P^2} e^{i\ell \cdot (X(z_+) - X(z_-)) + \frac{\ell^2}{2} \int e \omega_{+-}^2} \prod_{i=1}^n e^{i p_i \cdot X(z_i)} , \quad (3.26)$$

where the first term is from the action, the second from the gluing operator and the rest from the external vertex operators. As for tree-level amplitudes,  $X$  and  $e$  only appear linearly in the exponential and so act as Lagrange multipliers. As always, integrating out the zero-mode of  $X$  produces a momentum conserving  $\delta$ -function constraining  $\sum_i p_i = 0$ , while integrating out the non-zero modes of  $X$  freezes the quantum field  $P(z)$  to its classical value

$$P(z) = \ell \omega_{+-}(z) + \sum_{i=1}^n p_i \omega_{i*}(z) , \quad (3.27)$$

where  $\omega_{+-}(z) = \frac{dz(z_+ - z_-)}{(z - z_+)(z - z_-)}$ , in analogy to the one-forms  $\omega_{i*}$  defined in (1.26). Similarly, upon integrating out  $e$  we discover that the correlators have  $\delta$ -function support on the solutions to

$$P^2 = \ell^2 \omega_{+-}^2 . \quad (3.28)$$

Thus, inserting the gluing operator into the genus zero correlation function modifies the scattering equations to become the same ‘one-loop scattering equations’ one obtains [18] by localising the genus one correlation function on the boundary at  $q = 0$ .

Although equation (3.28) is required to hold at every point on the sphere, the holomorphic nature of  $P(z)$  actually makes (3.28) a finite dimensional constraint. Indeed, it is sufficient to pick a set of  $n - 1$  arbitrary points  $\{w_1, w_2, \dots, w_{n-1}\} \in \Sigma$  and require that

$$P^2(w_r) = \ell^2 \omega_{+-}^2(w_r) \quad (3.29)$$

at each of these points. Holomorphy of  $P(z)$  then ensures that (3.28) holds globally. It is worth pointing out that, even though the equations (3.29) individually depend on the choice of points  $w_r$ , their solutions do not. As a special case, one can take the  $w_r$  to coincide with some of the punctures, which, after multiplication with  $(w_r - z_i)$ , amounts to requiring

$$\text{SE}_i := \text{Res}_{z_i} \left\{ P^2(z) - \ell^2 \omega_{+-}^2(z) \right\} = 0 \quad (3.30)$$

for any  $n - 1$  of the  $n + 2$  punctures.

The remaining part of the correlation function comes from the two independent current algebras. To obtain the 1-loop amplitude in a particular colour ordering, we need to extract the coefficient of a given, single-trace contribution *by hand*. In addition, we must *by hand* only consider contributions in which  $z_+$  and  $z_-$  are adjacent in the colour ordering, ensuring that the colour ‘runs around the loop’. These conditions are analogous to the fact that we had to extract single-trace terms by hand even to obtain the tree amplitude. In both cases, they are symptomatic of the fact that this bosonic ambitwistor string model does not correctly describe pure bi-adjoint scalar theory. A further symptom of this sickness is that, as noted in [18, 19], the 1-loop scattering equations (3.28) contain certain ‘singular solutions’, where  $z_+ = z_-$ , corresponding to a tadpole. These solutions must again be discarded by hand. Later, we will investigate 1-loop amplitudes in gravity using a gluing operator in the Type II ambitwistor strings. In this case, the worldsheet CFT correctly generates the full answer in a consistent manner – in particular, in this consistent ambitwistor model, it will not be necessary to discard any terms by hand.

Altogether, the bosonic ambitwistor string worldsheet correlator of  $n$  vertex operators and the gluing operator leads to

$$\int \left\langle \Delta_{\phi^3}(z_+, z_-) \prod_{i=1}^n c(z_i) \tilde{c}(z_i) t_a^{(i)} j^a(z_i) \tilde{t}_{\tilde{a}}^{(i)} \tilde{j}^{\tilde{a}}(z_i) e^{ip \cdot X(z_i)} \right\rangle_{\alpha, \beta}$$

$$= \delta^d \left( \sum_i p_i \right) \sum_{\text{solns}} \int \frac{d^d \ell}{\ell^2} J^{-1} \text{PT}(\alpha) \text{PT}(\beta) , \quad (3.31)$$

when the external particles are in bi-colour structure  $(\alpha, \beta)$ . Here, the integral on the *lhs* is taken over the full moduli space, including both the moduli of the  $(n+2)$ -punctured Riemann sphere and the moduli of the gauge field  $e$ . On the *rhs*, the sum is taken over all non-singular solutions to the 1-loop scattering equations (3.28), weighted by the Jacobian

$$J = \omega_{rst}^2 \det \left( \frac{\partial \text{SE}_i}{\partial z_j} \right) \quad (3.32)$$

where each  $\omega_{rst} = S_{rs} S_{st} S_{tr}$  is the usual  $\text{vol SL}(2; \mathbb{C})$  factor from the zero modes of  $c$  and  $\tilde{c}$ ,  $\text{SE}_i$  is the  $i^{\text{th}}$  scattering equation (3.30) where  $i, j \in \{1, \dots, n+2\} / \{r, s, t\}$ . This agrees with the expression for the integrand of the 1-loop amplitude in this bi-adjoint theory given in [18, 19].

## 3.2 The Yang-Mills Gluing Operator at Tree Level

Having described in detail the method of gluing together correlators from two spheres using the gluing operator in coloured  $\phi^3$ , we now turn to the corresponding construction in Yang-Mills, whose amplitudes are given in the CHY framework by

$$\mathcal{A} = \int \left( \frac{1}{\omega_{123}} \right)^2 \prod_{i=4}^n \bar{\delta} \left( \text{Res}_{z_i} P^2 \right) \text{PT Pf}' \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \quad (3.33)$$

where the familiar Parke-Taylor factor is defined in the ordering  $1, \dots, n$  without loss of generality. The integrand also contains the reduced CHY YM-Pfaffian, as explained in eq. (1.37a).

The plan is again to take the scattering equation sitting at  $z_4$ , relax it and integrate by parts, and we expect to pick up contributions from many more boundaries of the moduli space. At this point, it is crucial to recall that the definition of the reduced Pfaffian  $\text{Pf}' M$  requires the choice of  $n-2$  distinguished points on the sphere, e.g.  $n-2$  of the external points and that the dependence on these points only drops out if all

the scattering equations hold. In other words, the quantity

$$S(z_i, z_j) \text{Pf} \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}^{\check{i}\check{j}}, \quad (3.34)$$

where the superscript  $\check{i}\check{j}$  indicates that the  $i^{\text{th}}$  and  $j^{\text{th}}$  row/column are to be removed before computing the Pfaffian, is *not* independent of the choice of  $i, j$  as soon as we go away from the solutions to the scattering equations.

In particular, this means that the contributions from the boundary in eq. (3.1) may individually depend on the choices of gauge fixing that we make prior to integrating by parts. The gauge invariance on the support of the scattering equations together with the global residue theorem, however, guarantee that, after summing over all contributions from the boundaries, the gauge invariance is restored and all the choices made drop out.

With this in mind let us choose to define “the” CHY Pfaffian away from the scattering equations as the one with the first and third column removed (i.e.  $i = 1, j = 3$ ). More precisely, we choose to remove two columns which are associated with particles that have no scattering equation and are non-adjacent in the colour ordering. As long as we are on the support of the scattering equations, we can make this choice without loss of generality, since it is really only a gauge choice, but as soon as we move away from the solutions of the scattering equations (and thus break gauge invariance) these choices start to matter. So our starting point for the integration by parts procedure is

$$\left(\frac{1}{\omega_{123}}\right)^2 \bar{\partial} \frac{1}{\text{Res}_{z_4} P^2} \prod_{i=5}^n \bar{\partial} (\text{Res}_{z_i} P^2) \text{PT}(\mathbb{I}) S(z_1, z_3) \text{Pf} \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}^{\check{1}\check{3}}. \quad (3.35)$$

Integrating by parts (or equivalently, using the global residue theorem) on this expression we find contributions from all the boundaries which have exactly two of the four punctures  $z_{1,2,3,4}$  on each side and cut<sup>4</sup> the colour ordering once or twice. The number of cuts distinguishes two types of channels that contribute to the glueing operator. The

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<sup>4</sup>The number of cuts of a colour order  $\alpha$  in a given channel, is defined as half the number of jumps from the left to the right sphere, and back, when stepping through the total set of external particles

single cut channel can be identified as a vector mode, while the double cut arises from an unphysical bi-adjoint scalar mode.

Thus we find that

$$\mathcal{A} = \sum_{\text{channels}} \bar{\mathcal{A}}_L^{\mu a} \Delta_{\mu\nu ab} \bar{\mathcal{A}}_R^{\nu b} + \bar{\mathcal{A}}_L^{ab} \Delta_{abcd} \bar{\mathcal{A}}_R^{cd} \quad (3.36)$$

where  $\bar{\mathcal{A}}_{L/R}$  denotes the building block, containing the external particles in the left/right subset as well as the node (c.f. fig. 3.1). They are contracted with the tensor structures

$$\Delta_{\mu\nu ab} = \frac{1}{\ell^2} \delta_{ab} \eta_{\mu\nu} , \quad \Delta_{abcd} = \frac{1}{\ell^2} \delta_{ac} \delta_{bd} . \quad (3.37)$$

Choosing for concreteness a channel where  $1, 2 \in L$  and  $3, 4 \in R$ , the building blocks are given by

$$\bar{\mathcal{A}}_L^{\mu a} = \int \left( \frac{1}{\omega_{1,2,*}} \right)^2 \prod_{i \in L \setminus \{1,2\}} \bar{\delta}(\text{Res}_{z_i} P_L^2) \text{PT}(\alpha_L)^a \text{Pf}' M_V^\mu \quad (3.38)$$

for the vector mode, and

$$\bar{\mathcal{A}}_L^{ab} = \int \left( \frac{1}{\omega_{1,2,*}} \right)^2 \prod_{i \in L \setminus \{1,2\}} \bar{\delta}(\text{Res}_{z_i} P_L^2) \text{PT}(\alpha_L)^{ab} \text{Pf} M_S \quad (3.39)$$

for the scalar mode. They are again supported on the ‘off-shell scattering equations’ and contain certain (reduced) Pfaffians.

The scalar mode Pfaffian is simply the original CHY Pfaffian, defined in terms of all  $n_{L/R}$  particles on the left/right. While this Pfaffian vanishes for a tree level configuration as a consequence of the scattering equations  $P^2 = 0$ , here it is easy to show that  $\text{Pf} M_S \propto \ell^2$ . This also ensures that the scalar mode does not contribute in any factorisation channel.

The new vector mode Pfaffian is given by

$$\text{Pf}' M_V^\mu \equiv \frac{\partial}{\partial(\varepsilon_*)_\mu} \text{Pf}' M_V \quad (3.40)$$

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from  $\alpha(i)$  to  $\alpha(i+1)$ . This number controls the divergence behaviour of the Parke-Taylor factor near a degeneration boundary.

where  $M$  is the  $(2n_{L/R} + 2)^{\times 2}$  anti-symmetric matrix

$$M_V = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \quad (3.41)$$

defined by the matrix entries

$$\begin{aligned} A_{i*} &= \left( p_i \cdot \ell + \frac{1}{2} \ell^2 \right) S(z_i, z_*) , & A_{12} &= \left( p_1 \cdot p_2 - \frac{1}{2} \ell^2 \right) S(z_1, z_2) , \\ B_{i*} &= \varepsilon_i \cdot \varepsilon_* S(z_i, z_*) , & B_{12} &= \varepsilon_1 \cdot \varepsilon_2 S(z_1, z_2) , \\ C_{i*} &= \varepsilon_i \cdot \ell S(z_i, z_*) , & C_{12} &= \varepsilon_1 \cdot p_2 S(z_1, z_2) , \\ C_{*i} &= \varepsilon_* \cdot \left( p_i + \frac{1}{2} \ell \right) S(z_*, z_i) , & C_{21} &= \varepsilon_2 \cdot p_1 S(z_2, z_1) , \\ C_{**} &= -\varepsilon_* \cdot \left( P - \ell \frac{\omega_{*1} + \omega_{*2}}{2} \right) (z_*) , & C_{11} &= -\varepsilon_1 \cdot P(z_1) , \\ & & C_{22} &= -\varepsilon_2 \cdot P(z_2) , \end{aligned} \quad (3.42)$$

for  $i = 1, 2$ , while all other matrix entries are defined by

$$\begin{aligned} A_{ij} &= p_i \cdot p_j S(z_i, z_j) , & B_{ij} &= \varepsilon_i \cdot \varepsilon_j S(z_i, z_j) , & C_{ij} &= \varepsilon_i \cdot p_j S(z_i, z_j) , \\ A_{ii} &= 0 , & B_{ii} &= 0 , & C_{ij} &= -\varepsilon_i \cdot P(z_i) , \end{aligned} \quad (3.43)$$

where  $i \neq j$ . Just as the original CHY matrix,  $M_V$  has a two dimensional kernel on the support of the scattering equations, spanned by the two sections

$$\gamma \in H^0(\Sigma, T^{1/2}) , \quad (3.44)$$

which means that it's Pfaffian  $\text{Pf } M$  vanishes. We compute it's canonically defined reduced Pfaffian by removing any two rows/columns from  $M$ , such that the resulting reduced matrix is of full rank, taking its Pfaffian and adjoining an appropriate Jacobian from the kernel, e.g.

$$\text{Pf}' M_V = S(z_i, z_j) \text{Pf}(M_V^{i,j}) \quad (3.45)$$

where the superscript indicates that the  $i^{\text{th}}$  and  $j^{\text{th}}$  row/column are to be removed. The scattering equations ensure that the reduced Pfaffian retains all the symmetry properties of the original Pfaffian, and is thus independent of the choice of which row/column to remove.

The building block (3.36) can be computed by inserting the standard vertex operators alongside one leg of the gluing operator, explicitly

$$\begin{aligned}\bar{\mathcal{A}}_L^{\mu a} &= \int \left\langle W_{12}(z_*) \mathcal{O}^{\mu a}(z_*) \prod_{i \in L \setminus \{1,2\}} \mathcal{O}_i(z_i) \right\rangle \Big|_{\alpha_L} \\ \bar{\mathcal{A}}_L^{ab} &= \int \left\langle W_{12}(z_*) \mathcal{O}^{ab}(z_*) \prod_{i \in L \setminus \{1,2\}} \mathcal{O}_i(z_i) \right\rangle \Big|_{\alpha_L}\end{aligned}\tag{3.46}$$

where  $\langle \dots \rangle|_{\alpha_L}$  is the instruction to pick out the appropriate trace structure. The vertex operators are defined as

$$\mathcal{O}^{\mu a} = c \tilde{c} j^a \delta(\gamma) \psi^\mu e^{i\ell \cdot X}, \quad \mathcal{O}^{ab} = c \tilde{c} j^a j^b e^{i\ell \cdot X}.\tag{3.47}$$

The factor  $W_{12}(z_*)$  contains the bosonic factor as in eq. (3.6), but the presence of the gauged fermionic current  $\mathcal{G} = P \cdot \psi$ , which satisfies  $\mathcal{G}^2 = P^2$  demands that the gluing operator also depends on the fermionic gauge field  $\chi$ . The correct choice is

$$W_{12}(z_*) = \exp \left( \frac{\ell^2}{2} \int_{\Sigma} e(x) \omega_{*1}(x) \omega_{*2}(x) + \frac{\ell^2}{2} \int_{\Sigma \times \Sigma} \chi(x) \omega_{*1}(x) S(x, y) \chi(y) \omega_{*2}(y) \right).\tag{3.48}$$

The need to add an unphysical (non-propagating) scalar mode is mysterious, but since there is currently no consistent ambitwistor model for Yang-Mills theory, it is perhaps not too surprising. The understanding of this contribution is still work in progress.

We have checked this construction numerically up to seven points.

### 3.3 The Yang-Mills Gluing Operator at Loop Level

In this section, we will concentrate on the case of 1-loop amplitudes. The main new ingredient here is that the worldsheet fermion system  $\psi$  requires a choice of spin structure. Thus, the gluing operator must account for states in both the Neveu-Schwarz and Ramond sectors, allowing both gluons and gluinos to run around the loop. Consequently, the gluing operator in Yang-Mills theory takes the form

$$\Delta_{\text{YM}}(z_+, z_-) = \int \frac{d^d \ell}{\ell^2} (\Delta_{\text{NS}}(z_+, z_-) + \Delta_{\text{R}}(z_+, z_-)) c \tilde{c} j^a(z_+) \delta_{ab} c \tilde{c} j^b(z_-) W_{\text{YM}}(z_+, z_-)\tag{3.49}$$

where  $\Delta_{\text{NS}}$  and  $\Delta_{\text{R}}$  are the contributions from the NS and R sectors, respectively. We will see that the worldsheet correlation function involving  $n$  external gluon vertex operators as well as this gluing operator can be computed in closed form in both the NS and R sectors.

The factor  $W_{YM}$  again compensates the BRST transformation of the local insertions and depends on both bosonic and fermionic gauge fields. The correct choice for  $W_{YM}$  is given by

$$W_{YM}(z_+, z_-) = \exp \left( \frac{\ell^2}{2} \int_{\Sigma} e(x) \omega_{+-}^2(x) + \frac{\ell^2}{2} \int_{\Sigma \times \Sigma} \chi(x) \omega_{+-}(x) S(x, y) \chi(y) \omega_{+-}(y) \right). \quad (3.50)$$

This factor is common to both the NS and R sectors. (There will be further  $\chi$  dependence in part of the gluing operator specific to the NS sectors.)

We remark that, just as in the bi-adjoint scalar theory, both  $\Delta_{\text{NS}}$  and  $\Delta_{\text{R}}$  depend on the field  $X$  only through a factor  $e^{i\ell \cdot (X(z_+) - X(z_-))}$ . Thus, integrating out  $X$  again leads to

$$P(z) = \ell \omega_{+-}(z) + \sum_{i=1}^n p_i \omega_{i*}(z), \quad (3.51)$$

and then integrating out the bosonic gauge field  $e$  leads to the same 1-loop scattering equations

$$P^2 = \ell^2 \omega_{+-}^2 \quad (3.52)$$

as before. Again, these are the scattering equations that [18] required to describe 1-loop amplitudes in SYM theory.

Similarly, the only dependence that the YM gluing operator has on the current algebra associated to the target space gauge group is the common factor of  $j^a(z_+) \delta_{ab} j^b(z_-)$ . The current algebra correlator factors out of the correlation function and generates a sum over all possible Parke-Taylor factors in all orders, including multi-trace terms. As with the bi-adjoint scalar theory discussed above, we *by hand* choose to extract only those single-trace terms in which  $z_+$  and  $z_-$  are adjacent in the colour ordering. Again, we expect that a fully consistent ambitwistor string for Einstein-Yang-Mills would possess a gluing operator that does not need such manipulation<sup>5</sup>.

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<sup>5</sup>It would also be interesting to investigate this in the context of the variant of ambitwistor string proposed in [31] that describes a theory in which the Yang-Mills action is replaced by  $\int \text{tr}(d_A b \wedge *F)$ , where  $b$  is an adjoint-valued 1-form, independent of the connection  $A$ .



### 3.3.1 Neveu-Schwarz sector

We now describe the NS sector part of the gluing operator (3.49). In this sector,

$$\Delta_{\text{NS}}(z_+, z_-) = \mathcal{O}^\mu(z_+) \Delta_{\mu\nu} \mathcal{O}^\nu(z_-), \quad (3.53)$$

where

$$\mathcal{O}^\mu(z_\pm) = \delta(\gamma(z_\pm)) \left( \psi^\mu(z_\pm) - \ell^\mu \int_\Sigma \chi(x) S(z_\pm, x) \omega_{+-}(x) \right) e^{\pm i\ell \cdot X(z_\pm)} \quad (3.54a)$$

describes the contribution of an off-shell vector mode, and the tensor structure is

$$\Delta_{\mu\nu} = \eta_{\mu\nu} - \frac{\ell_\mu \xi_\nu + \xi_\mu \ell_\nu}{\xi \cdot \ell}. \quad (3.54b)$$

Here  $\xi_\mu$  is an arbitrary vector that ensures only transverse modes propagate around the loop. One may check that the full gluing operator (3.49) (including the factor of  $W(z_+, z_-)$ ) is BRST invariant under the transformations (1.31)<sup>6</sup>. This BRST invariance, in particular, ensures that the amplitude is independent of the particular choice of  $\xi$ .

We now consider the path integral over the fermionic fields, which will yield a (reduced) CHY Pfaffian for  $n + 2$  particles. The gluing operator depends on  $\chi$  through  $\mathcal{O}^\mu$  as well as deforming the fermionic moduli through  $W(z_+, z_-)$ . Thus, as well as modifying the scattering equations as before (through its dependence on  $e$ ), the insertion of this Yang-Mills gluing operator will also change the entries of the  $A$  and  $C$  block of the CHY Pfaffian.

Putting all this together, in a generic basis, the correlator is given by

$$\begin{aligned} \Delta_{\mu\nu} \left\langle \right. & \delta(\gamma(z_+)) \left( \psi^\mu(z_+) - \ell^\mu \int \chi_0(w_+) \omega_{+-}(w_+) S(w_+, z_+) \right) \\ & \delta(\gamma(z_-)) \left( \psi^\nu(z_-) - \ell^\nu \int \chi_0(w_-) \omega_{+-}(w_-) S(w_-, z_-) \right) \\ & \exp \left( \frac{\ell^2}{2} \int_{\Sigma \times \Sigma} \chi_0(y) \omega_{+-}(y) S(y, y') \chi_0(y') \omega_{+-}(y') \right) \\ & \left. \prod_{i=1}^n \delta(\gamma(z_i)) \varepsilon_i \cdot \psi(z_i) \prod_{r=1}^n \delta(\beta(x_r)) P(x_r) \cdot \psi(x_r) \right\rangle_{\beta\gamma, \psi, \chi_0}, \end{aligned} \quad (3.55)$$

<sup>6</sup>In fact, there is a potential failure of BRST invariance in the NS sector arising on the boundary of the moduli space where  $z_+ = z_-$ . We will see later that this failure is cancelled by the R sector.

where the first three lines contain the relevant parts of the gluing operator and the final line contains the external vertex operators as well as the picture changing operators. Recall that the field  $P(z)$  is frozen to its classical value eq. (3.51). If not for the fact that the  $\psi$  fields at  $z_{\pm}$  are contracted into the target vector propagator  $\Delta_{\mu\nu}(\ell)$ , this would readily evaluate to a simple Pfaffian. This motivates us to rewrite

$$\Delta_{\mu\nu} \psi^{\mu}(z_{+}) \psi^{\nu}(z_{-}) = \Delta_{\mu\nu} \frac{\partial^2}{\partial \epsilon_{\mu}^{+} \partial \epsilon_{\nu}^{-}} \varepsilon^{+} \cdot \psi(z_{+}) \varepsilon^{-} \cdot \psi(z_{-}) , \quad (3.56)$$

where  $\varepsilon^{\pm}$  are auxiliary ‘polarisation vectors’ for the state flowing in the propagator – note that, as usual, these off-shell polarisation vectors are not required to be transverse to  $\ell$ . With this replacement, we can immediately evaluate the correlator (3.55) to

$$\frac{V(\{z_i\}_{i=1}^{n,+,-} | \{x_r\}_{r=1}^n)}{V(\{z_i\}_{i=1}^{n,+,-}) V(\{x_r\}_{r=1}^n)} \Delta_{\mu\nu} \frac{\partial^2}{\partial \varepsilon_{\mu}^{+} \partial \varepsilon_{\nu}^{-}} \text{Pf} \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} , \quad (3.57)$$

with the familiar Vandermonde factors

$$\begin{aligned} V(\{z_i\}_{i=1}^{n,+,-}) &= \prod_{i < j=1}^{n,+,-} \frac{1}{S(z_i, z_j)} , & V(\{x_r\}_{r=1}^n) &= \prod_{r < s=1}^n \frac{1}{S(x_r, x_s)} , \\ V(\{z_i\}_{i=1}^{n,+,-} | \{x_r\}_{r=1}^n) &= \prod_{i=1}^{n,+,-} \prod_{r=1}^n \frac{1}{S(z_i, x_r)} , \end{aligned} \quad (3.58)$$

and the matrix entries

$$\begin{aligned} A_{rs} &= \left( P(x_r) \cdot P(x_s) - \frac{\ell^2}{2} \omega_{+-}(x_r) \omega_{+-}(x_s) \right) S(x_r, x_s) , \\ B_{ij} &= \varepsilon_i \cdot \varepsilon_j S(z_i, z_j) , & C_{ir} &= \varepsilon_i \cdot (P(x_r) - \ell \omega_{+-}(x_r)) S(z_i, x_r) , \end{aligned} \quad (3.59)$$

for  $r, s = 1, \dots, n$  and  $i, j = 1, \dots, n, +, -$ . It is readily shown that (3.57) is independent of the locations  $x_r$ , by checking that it is free of poles as any  $x_r \rightarrow x_s$  as well as  $x_r \rightarrow z_i$ , on the support of the scattering equations (3.52), and appealing to Liouville’s theorem.

To simplify computations from here onwards, we choose to work in a picture where the  $n$  external NS vertex operators are at picture zero, where the two zero modes of the  $\gamma$  ghosts at genus zero are absorbed by the  $\delta(\gamma)$  factors of the gluing operator. More explicitly, this statements means that we choose to expand the fermionic moduli

in a basis

$$\chi(z) = \sum_{r=1}^{(n+2)-2} \eta_r \chi_r(z) , \quad (3.60)$$

where for  $r = 1, \dots, n$ , the  $\eta_r$  are Grassmann valued constants, transforming as an element of  $T_{z_r}^{1/2}$ , and the moduli

$$\text{span}(\{\chi_r\}_{r=1}^n) = H^1(\Sigma, T_\Sigma^{1/2}(-z_1 - \dots - z_n - z_+ - z_-))$$

are chosen to extract the residue at the  $r^{\text{th}}$  marked point. That is, for any meromorphic  $f \in H^0(\Sigma, K_\Sigma^{3/2}(z_1 + \dots + z_n + z_+ + z_-))$ , we have

$$\int_\Sigma \chi_r(z) f(z) = \text{Res}_r f(z) , \quad (3.61)$$

where the residue is understood as an element of  $K^{1/2}|_{z_r}$ . (This pairing is an instance of Serre duality.) Note that there are exactly  $n$  such moduli, two less than the total number of points marked by either the external vertex operators or the gluing operator. Thus, after performing the  $\beta\gamma$  path integral, our choice of basis amounts to descending all  $n$  external vertex operators to picture 0. In addition, since  $S(z_\pm, x) \omega_{+-}(x)$  has no pole as  $x$  approaches any of the external marked points (at least for generic choices of these points), we have

$$\int_\Sigma \chi_r(x) S(z_\pm, x) \omega_{+-}(x) = 0 \quad \forall r \in \{1, \dots, n\} \quad (3.62)$$

so with this choice of basis, both the shift of  $\psi^\mu$  in (3.54a) and the fermionic contribution to  $W(z_+, z_-)$  in (3.50) vanish. Consequently, in this basis the contribution from the  $\beta\gamma$  and  $\psi$  system becomes

$$\Delta_{\mu\nu} \left\langle \delta(\gamma(z_+)) \psi^\mu(z_+) \delta(\gamma(z_-)) \psi^\nu(z_-) \prod_{i=1}^n (\varepsilon_i \cdot P(z_i) + \varepsilon_i \cdot \psi(z_i) p_i \cdot \psi(z_i)) \right\rangle_{\beta\gamma, \psi} \quad (3.63)$$

where we recall that  $P(z_i)$  is frozen by (3.27). We stress that the expression (3.63) is simply the limit of (3.55) where one of each of the  $x_r$  is taken to coincide with one of each of the  $z_i$ .

We now proceed to evaluate the correlator (3.63). The  $\beta\gamma$  path integral is trivially performed and gives a factor

$$\langle \delta(\gamma(z_+)) \delta(\gamma(z_-)) \rangle_{\beta\gamma} = \frac{\sqrt{dz_+ dz_-}}{z_+ - z_-} =: S_{+-} . \quad (3.64)$$

The more interesting part is the  $\psi$  path integral. Including the  $\beta\gamma$  contribution (3.64), we find

$$S_{+-} \Delta_{\mu\nu} \frac{\partial^2}{\partial \varepsilon_\mu^+ \partial \varepsilon_\nu^-} \text{Pf} \left( \begin{array}{cc|c} 0 & \varepsilon^+ \cdot \varepsilon^- & S_{+-} & \varepsilon^+ \cdot v_j & S_{+j} \\ \cdot & & 0 & \varepsilon^- \cdot v_j & S_{-j} \\ \hline \cdot & & & & M \end{array} \right), \quad (3.65)$$

where  $M$  is the full,  $(2n)^{\times 2}$ , tree level CHY matrix involving only the external particles, and  $v_j$  represents either  $p_j$  or  $\varepsilon_j$  depending on whether the index  $j$  is in the first of second half of  $M$ . We also abbreviated  $S_{ij} \equiv S(z_i, z_j)$ , and all indices are understood modulo  $n$  unless otherwise stated.

We highlight again that the expression (3.65) is simply the limit of (3.57) where one of each of the  $x_r$  is taken to coincide with one of each of the  $z_i$ .

To perform the derivatives, we use that the variation of a Pfaffian is given by

$$\delta \text{Pf}(M) = \sum_{i < j} (-1)^{i+j+1} \delta M_{ij} \text{Pf}(M^{ij}) \quad (3.66)$$

where  $M^{ij}$  denotes the matrix obtained by removing rows/columns  $i, j$ . We find that the correlator (3.63) becomes

$$S_{+-}^2 \left( \text{tr}(\Delta) + \sum_{i < j} v_i^\mu \Delta_{\mu\nu} v_j^\nu \frac{S_{i+} S_{-j} - S_{i-} S_{+j}}{S_{+-}} \frac{\partial}{\partial M_{ij}} \right) \text{Pf}(M), \quad (3.67)$$

where  $\Delta_\mu^\mu = d - 2 = 8$  is the number of transverse polarisation states in this NS sector.

To compare to the result of [18, 19], we must process it further. Using the  $q$ -expansion of the Szegő kernel given in eq. (B.1) we find that, on the support of the 1-loop scattering equations (3.28),

$$\text{Pf}(M) = \text{Pf}(M_3)|_{q^0} \quad (3.68)$$

so that the Pfaffian of the full,  $2n \times 2n$  tree-level CHY matrix involving only the external states<sup>7</sup> precisely agrees with the  $\mathcal{O}(q^0)$  term in the expansion of the spin-structure 3 Pfaffian on the torus, expanded around the degeneration limit  $q = 0$ . Thus, the first term of (3.67) can immediately be written in terms of an object appearing in the

<sup>7</sup>Note that this Pfaffian does not vanish, since we are on the support of the 1-loop scattering equations  $P^2(z) = \ell^2 \omega_{+-}^2(z)$ .

genus one result, localised to  $q = 0$ . Furthermore, a straightforward (though somewhat tedious) calculation reveals that the contribution to (3.67) coming from the part of  $\Delta_{\mu\nu}$  that is proportional to the metric  $\eta_{\mu\nu}$  yields

$$\sum_{i < j} \frac{S_{i+} S_{-j} - S_{i-} S_{+j}}{S_{+-}} v_i \cdot v_j \frac{\partial}{\partial M_{ij}} \text{Pf}(M) = \text{Pf}(M_3)|_{\sqrt{q}}, \quad (3.69)$$

*i.e.*, the co-efficient of  $\sqrt{q}$  in the  $q$ -expansion of the same Pfaffian as the torus degenerates.

Combining these two, our result (3.67) for the genus zero correlator including the NS sector part of the gluing operator can be written as

$$S_{+-}^2 \left( 8 \text{Pf}(M_3)|_{q^0} + \text{Pf}(M_3)|_{\sqrt{q}} \right) - S_{+-}^2 [\eta_{\mu\nu} - \Delta_{\mu\nu}] \sum_{i < j} v_i^\mu v_j^\nu (S_{i+} S_{-j} - S_{i-} S_{+j}) \frac{\partial}{\partial M_{ij}} \text{Pf}(M). \quad (3.70)$$

When multiplied by the appropriate current correlator (worldsheet Parke-Taylor factor), the first line here agrees precisely with the result of [18, 19] for the NS sector contribution to the 1-loop integrand of SYM. We would thus like to show that the second line vanishes.

Firstly, note that the tensor structure

$$\eta_{\mu\nu} - \Delta_{\mu\nu} = \frac{\ell_\mu \xi_\nu + \xi_\mu \ell_\nu}{\ell \cdot \xi}$$

reveals that the second line in (3.70) comes from longitudinal states flowing through the propagator. To see that these states decouple, so that this line in fact vanishes, we rewrite it in the equivalent form

$$\frac{1}{\ell \cdot \xi} S_{+-}^2 \left[ \text{Pf} \left( \begin{array}{cc|c} 0 & 0 & \ell \cdot v_j S_{+j} \\ \cdot & 0 & \xi \cdot v_j S_{-j} \\ \hline \cdot & \cdot & M \end{array} \right) + \text{Pf} \left( \begin{array}{cc|c} 0 & 0 & \xi \cdot v_j S_{+j} \\ \cdot & 0 & \ell \cdot v_j S_{-j} \\ \hline \cdot & \cdot & M \end{array} \right) \right]. \quad (3.71)$$

It is easily verified that each of these two Pfaffians vanishes on the support of the 1-loop scattering equations (3.28), because e.g. the vector  $V_j = (S_{+-}^{-1}, 0 | S_{j-}^{-1}, 0_j)^T$  is in the kernel of the first matrix, while  $U_j = (0, S_{-+}^{-1} | S_{j+}^{-1}, 0_j)^T$  similarly lies in the kernel

of the other. We emphasise that this *does not* mean that the tensor structure in the gluing operator may simply be taken to be  $\eta_{\mu\nu}$ , since any choice of  $\xi$  removes the two longitudinal modes. This can be seen explicitly in the factor of  $8 = d - 2$  in the first line of (3.70).

One can also see this decoupling of the two unphysical degrees of freedom directly at the level of the vertex operators. Let  $\xi$  and  $\xi'$  be two different choices of vector used to define the longitudinal part of the tensor structure of the propagator. Then

$$\Delta_{\mu\nu}^{(\xi)} - \Delta_{\mu\nu}^{(\xi')} = 2 \left( \frac{\ell_{(\mu} \xi'_{\nu)}}{\ell \cdot \xi'} - \frac{\ell_{(\mu} \xi_{\nu)}}{\ell \cdot \xi} \right) \quad (3.72)$$

Thus, the difference between gluing operators defined with the help of  $\xi$  and  $\xi'$  is proportional to

$$\delta(\gamma) \ell \cdot \psi_+ e^{i\ell \cdot X_+} - \delta(\gamma) [(\xi \cdot \ell)(\xi' \cdot \psi_-) - (\xi' \cdot \ell)(\xi \cdot \psi_-)] e^{-i\ell \cdot X_-} + (z_+ \leftrightarrow z_-) . \quad (3.73)$$

The insertion at  $z_+$  can be written as a BRST variation

$$\delta(\gamma) \ell \cdot \psi e^{i\ell \cdot X} = Q \circ \left( \delta'(\gamma) e^{i\ell \cdot X} \right) , \quad (3.74)$$

where only the fermionic part of the BRST operator contributes non-trivially. However, in the difference of propagators with different longitudinal parts, the insertion at  $z_-$  is actually BRST closed, since

$$\begin{aligned} Q \circ \left( \delta(\gamma) [(\xi \cdot \ell)(\xi' \cdot \psi) - (\xi' \cdot \ell)(\xi \cdot \psi)] e^{-i\ell \cdot X} \right) \\ = -\delta(\gamma) \partial\gamma [(\xi \cdot \ell)(\xi' \cdot \ell) - (\xi' \cdot \ell)(\xi \cdot \ell)] e^{-i\ell \cdot X} = 0 , \end{aligned} \quad (3.75)$$

where the second line comes from double contractions, the factor of  $\delta(\gamma)$  in the insertion annihilating the factor of  $\gamma$  in the BRST operator at lowest order. Consequently, the difference between gluing operators with two different choices of longitudinal part is BRST exact, and decouples from any correlation function involving on-shell external states. Again, this does not mean that the tensor structure in the gluing operator may be taken to be  $\eta_{\mu\nu}$ .

To summarise, we have shown that the operator insertion eq. (3.53) on the sphere gives rise to the NS sector contribution to the one-loop SYM integrand found by [18, 19]. In particular, we computed the CFT correlator with any number of gluon vertex operator insertions and demonstrated that the longitudinal degrees of freedom through the node decouple on the support of the scattering equations.

### 3.3.2 Ramond sector

As noted above, the  $\psi$  system also has an associated Ramond sector that provides spin field vertex operators (1.41b) which describe gluinos. To account for the possibility that these gluinos run around the loop, the gluing operator also contains a contribution from the Ramond sector, to which we now turn.

As a first attempt, one might assume that Ramond sector contribution to the SYM gluing operator should simply be

$$\Delta_{\text{R}}(z_+, z_-) \stackrel{?}{=} e^{-\phi/2} \Theta_{\alpha}(z_+) e^{i\ell \cdot X(z_+)} \left(\frac{1}{2} C^{-1} \bar{\ell}\right)^{\alpha\beta} e^{-\phi/2} \Theta_{\beta}(z_-) e^{-i\ell \cdot X(z_-)} \quad (3.76)$$

which is a sum over the corresponding off-shell continuation  $e^{-\phi/2} \Theta_{\alpha} e^{i\ell \cdot X}$  of the Ramond sector vertex operators inserted at  $z_{\pm}$ , joined by the tensor structure of the spin-1/2 propagator  $\frac{1}{2} (C^{-1} \bar{\ell})^{\alpha\beta}$ .

However, while (3.76) is correct, as it stands this operator cannot be added to the NS part of the propagator due to a mismatch in picture number – the operator (3.76) has picture number  $2 \times (-1/2) = -1$ , while the NS part has  $2 \times (-1) = -2$ . Hence we change picture on one leg and write the Ramond sector contribution instead as

$$\Delta_{\text{R}}(z_+, z_-) = e^{-\phi/2} \Theta_{\alpha}(z_+) e^{i\ell \cdot X(z_+)} (C^{-1})^{\alpha}_{\dot{\beta}} e^{-3\phi/2} \Theta^{\dot{\beta}}(z_-) e^{-i\ell \cdot X(z_-)} , \quad (3.77)$$

which has picture number  $-1/2 - 3/2 = -2$  and thus carries the right quantum numbers to be added to the NS contribution. One can check that both insertions (3.76) and (3.77) are indeed BRST closed, despite  $\ell^2 \neq 0$ , using the action (1.31) and the

OPEs among spin fields and fermions<sup>8</sup>

$$\begin{aligned}
\Theta_\alpha(z) \Theta_\beta(w) &\sim S(z, w)^{1/4} \sqrt{\omega_{w*}(z)} (\gamma^\mu C)_{\alpha\beta} \eta_{\mu\nu} \psi^\nu(w) \\
\Theta^{\dot{\alpha}}(z) \Theta_\beta(w) &\sim S(z, w)^{5/4} C^{\dot{\alpha}}_{\beta} \\
\psi^\mu(z) \Theta_\alpha(w) &\sim \sqrt{\omega_{w*}(z)} \gamma^\mu_{\alpha\dot{\beta}} \Theta^{\dot{\beta}}(w) \\
\psi^\mu(z) \Theta^{\dot{\alpha}}(w) &\sim \sqrt{\omega_{w*}(z)} (\bar{\gamma}^\mu)^{\dot{\alpha}\beta} \Theta_\beta(w) \\
\psi^\mu(z) \psi^\nu(w) &\sim S(z, w) \eta^{\mu\nu} \\
e^{q\phi(z)} e^{q'\phi(w)} &\sim (z-w)^{-qq'} e^{(q+q')\phi(w)} .
\end{aligned} \tag{3.78}$$

which are the same as in usual string theory. In principle, these may be used to evaluate any worldsheet correlator in the Ramond sector. There are analogous expressions for the tilded spin fields, which also carry *holomorphic* conformal weight.

It is important to note however that the NS and R components of the glueing operator separately have a BRST anomaly arising on the boundary of the moduli space where  $z_+ = z_-$ . To understand this, we consider the limit of the glueing operator as  $z_+ \rightarrow z_-$ : using the OPE of the fields involved we find the insertion

$$\lim_{z_- \rightarrow z_+} \Delta_{\text{YM}}(z_+, z_-) = \int \frac{d^{10}\ell}{\ell^2} c \partial c \tilde{c} \partial \tilde{c} \delta(\gamma) \delta(\partial\gamma) (8-8) \tag{3.79}$$

at the point where the two nodes meet, with the two contributions of opposite sign coming from the NS and R<sup>9</sup> sector respectively and we fixed  $d = 10$ . In addition, the entire correlator comes multiplied by the cyclic sum over Parke-Taylor factors, whose limit,

$$\begin{aligned}
&\lim_{z_- \rightarrow z_+} \sum_{i=1}^n \text{PT}(\{1, \dots, i, +, -, i+1, \dots, n\}) \\
&= -\frac{1}{2} \sum_{i=1}^n \omega_{i, i+1}(z_+) \text{PT}(\{1, \dots, i, +, i+1, \dots, n\}) ,
\end{aligned} \tag{3.80}$$

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<sup>8</sup>We adopt the conventions of Wess & Bagger for 10d chiral Dirac matrices

$$\gamma^\mu_{\alpha\dot{\beta}}, \quad (\bar{\gamma}^\mu)^{\dot{\alpha}\beta}, \quad C^{\dot{\alpha}}_{\beta} = -(C^T)_{\beta}^{\dot{\alpha}}, \quad (C^{-1})^{\alpha}_{\dot{\beta}} = -(C^{-1T})^{\alpha}_{\dot{\beta}}$$

and the chiral spin fields  $\Theta_\alpha(z), \Theta^{\dot{\alpha}}(z)$ . Further, we use the common notation  $p_{\alpha\dot{\beta}} = p_\mu \gamma^\mu_{\alpha\dot{\beta}}$  and  $\bar{p}^{\dot{\alpha}\beta} = p_\mu (\bar{\gamma}^\mu)^{\dot{\alpha}\beta}$  for mapping a vector to a bispinor. The Dirac matrices have a number of important properties, for instance

$$(pC)^T = -pC \quad \text{and} \quad (C^{-1}\bar{p})^T = -C^{-1}\bar{p} .$$

These properties will be useful later.

<sup>9</sup>This uses the bosonisation identity  $e^{-2\phi} = \delta(\gamma)\delta(\partial\gamma)$ .



is finite due to the  $U(1)$  decoupling identity. Notice that in this limit all dependence on the loop momentum  $\ell$  drops out, apart from the overall prefactor.

The operator insertion (3.79) might seem innocuous, but is actually dangerous, since it causes the bases that were used for the moduli of the gauge fields to become degenerate. Conversely, there are directions in the moduli space, of which the integrand is independent. This can be seen by considering that the integrand depends on the moduli only through the combinations

$$\int_{\Sigma} e_0 P^2 \quad \text{and} \quad \int_{\Sigma} \chi_0 \psi \cdot P, \quad (3.81)$$

and noticing that  $P^2$  and  $\psi \cdot P$  are now finite at  $z_+$ . For instance, in this limit we have

$$\int_{\Sigma} e_0 P^2 = \int_{\Sigma} e_0 \left( \sum_{i \neq j} p_i \cdot p_j \omega_{i*} \omega_{j*} \right), \quad (3.82)$$

which is clearly finite at  $z_+$  and has also become independent of  $\ell$ . To see that this pairing is now degenerate, we expand the field  $e_0$  in a basis of some coordinates on the  $n - 1$  dimensional moduli space, e.g.

$$e(z) = \sum_{r=1}^{n-1} m^r e_r(z) \quad \text{with} \quad \int_{\Sigma} e_r P^2 = P^2(w_r). \quad (3.83)$$

It is then easy to show that there are two linear combinations of the coordinates  $m^r$  which drop out. The situation is completely analogous for the fermionic moduli. The integral over the moduli which the integrand is independent of produces a  $0^2$  for the moduli of  $\chi$ , while those of  $e$  contribute  $\infty^2$ . This can be regularised and the net contribution made finite, though potentially ambiguous.

This anomaly is particularly problematic since  $z_+ = z_-$  actually always arises in  $(n - 2)!$  of the solutions to the scattering equations, called ‘singular solutions’. One way to handle this is to discard certain singular solutions of the scattering equations, which effectively regularises the operator insertion and is discussed in [15]. If however both NS and R parts are included, as in (3.79), the BRST ambiguity cancels between the sectors. This cancellation depends crucially on the relative coefficient between the NS and R term and is a manifestation of the target space supersymmetry of the model.

We now compute the correlator of  $n$  external NS sector vertex operators in picture 0, together with this R sector contribution to the gluing operator. This can be done using the remarkable, closed-form expression for the spin field correlator in the  $\psi$  path integral obtained for example by Haertl, Schlotterer and Stieberger in [157, 158]. They

find

$$\begin{aligned}
& (C^{-1})^\alpha_{\dot{\beta}} \left\langle \Theta_\alpha(z_+) \Theta^{\dot{\beta}}(z_-) \prod_{i=1}^{2n} v_i \cdot \psi(z_i) \right\rangle_\psi \\
&= S(z_+, z_-)^{5/4} \prod_{i=1}^{2n} \sqrt{\omega_{+-}(z_i)} \sum_{m=0}^n 2^{-m} \sum_{\rho \in S_{2n}/\mathcal{Q}_{n+1,m}} \text{sgn}(\rho) \text{tr}(v_{\rho(1)} \bar{v}_{\rho(2)} \cdots \bar{v}_{\rho(2m)}) \quad (3.84) \\
&\times \prod_{j=1}^{n-m} v_{\rho(2m+2j-1)} \cdot v_{\rho(2m+2j)} \frac{S(z_{\rho(2m+2j-1)}, z_{\rho(2m+2j)}) S(z_+, z_-)}{S(z_{\rho(2m+2j-1)}, z_+) S(z_{\rho(2m+2j)}, z_-)},
\end{aligned}$$

where the summation is taken over permutations

$$\begin{aligned}
S_{2n}/\mathcal{Q}_{n+1,m} \equiv \{ \rho \in S_{2n} : & \rho(1) < \rho(2) < \cdots < \rho(2m), \\
& \rho(2m+2j-1) < \rho(2m+2j) \ \forall j = 1, 2, \dots, n-m, \\
& \rho(2m+2) < \rho(2m+4) < \cdots < \rho(2n) \} . \quad (3.85)
\end{aligned}$$

To avoid possible confusion we emphasise that in this formula there are  $2n$  distinct marked points  $z_i$  and vectors  $v_i$ , which we only later specialise to the particular configuration we have. Reference [157, 158] actually treats the more general case of arbitrary genus, and where the spinor indices on  $\Theta_\alpha$  and  $\Theta^{\dot{\beta}}$  are left uncontracted. In our case these indices are joined using  $C^{-1}$ , which allows us to make considerable simplifications in (3.84).

To begin, note that permutations in  $S_{2n}/\mathcal{Q}_{n+1,m}$  which leave the first  $2m$  indices fixed act trivially on  $\text{tr}(v_{\rho(1)} \bar{v}_{\rho(2)} \cdots \bar{v}_{\rho(2m)})$ , and that the coefficient of this trace sums to give a Pfaffian. Thus the inner sum in (3.84) becomes

$$\begin{aligned}
& \sum_{\rho \in S_{2n}/\mathcal{Q}_{n+1,m}} \text{sgn}(\rho) \text{tr}(v_{\rho(1)} \bar{v}_{\rho(2)} \cdots \bar{v}_{\rho(2m)}) \\
& \times \prod_{j=1}^{n-m} v_{\rho(2m+2j-1)} \cdot v_{\rho(2m+2j)} \frac{S(z_{\rho(2m+2j-1)}, z_{\rho(2m+2j)}) S(z_+, z_-)}{S(z_{\rho(2m+2j-1)}, z_+) S(z_{\rho(2m+2j)}, z_-)} \quad (3.86) \\
&= \sum_{\substack{\mathbf{b} \subset \{1, \dots, 2n\} \\ |\mathbf{b}|=2m}} \text{tr}(v_{i_1} \bar{v}_{i_2} \cdots \bar{v}_{i_{2m}})_{i \in \mathbf{b}} \text{Pf} \left( v_i \cdot v_j \frac{S_{ij} S_{+-}}{S_{\min(i,j),+} S_{\max(i,j),-}} \right)_{i,j \in \mathbf{b}^c}
\end{aligned}$$

where the sum in the final line runs over all ordered subsets  $\mathbf{b}$  of the complete set of indices  $\{1, \dots, 2n\}$  of length  $2m$ .

We can simplify much further still by writing

$$\text{tr}(v_1 \bar{v}_2 v_3 \cdots \bar{v}_{2n}) = \frac{1}{2} \text{tr}(\not{v}_1 \not{v}_2 \cdots \not{v}_{2n}) + \frac{1}{2} \text{tr}(\not{v}_1 \not{v}_2 \cdots \not{v}_{2n} \Gamma_{d+1}) \quad (3.87)$$

and consider the contribution of the two terms separately. Due to the fact that the Clifford algebra is a representation of a fermionic QM system, we have the beautiful identities<sup>10</sup> for the vector part

$$\begin{aligned} \text{tr}(\phi_1 \cdots \phi_{2n}) &= \text{tr}(1) \text{Pf}(V) , \\ V_{ij} &= v_i \cdot v_j \text{sgn}(i - j) , \end{aligned} \tag{3.88a}$$

and the axial part

$$\begin{aligned} \text{tr}(\phi_1 \cdots \phi_{2n} \Gamma_{d+1}) &= \frac{\text{tr}(1)}{(d-1)!!} \int d^d \Psi_0 \text{Pf}(A) , \\ A_{ij} &= v_i \cdot v_j \text{sgn}(i - j) + v_i \cdot \Psi_0 v_j \cdot \Psi_0 , \end{aligned} \tag{3.88b}$$

of the trace of  $2n$  gamma-matrices, where  $\text{tr}(1) = 2^{d/2} = 32$ . Here  $\Psi_0^\mu$  is a set of  $d$  Grassmann numbers. Both these identities may be derived by computing a fermionic QM path integral on the circle (or simply verified using the Clifford algebra, Berezin integration rules and recursive definition of the Pfaffian).

The fact that both factors in (3.86) can be written in terms of Pfaffians allows us to combine them. Recall that the Pfaffian of the sum of any two antisymmetric matrices  $X, Y$  can be expressed as<sup>11</sup>

$$\text{Pf}(X + Y) = \sum_{\mathbf{b} \in \substack{\text{ordered} \\ \text{subsets}}} \text{sgn}(\mathbf{b}, \mathbf{b}^c) \text{Pf}(X_{ij})_{ij \in \mathbf{b}} \text{Pf}(Y_{ij})_{ij \in \mathbf{b}^c} , \tag{3.89}$$

where the sum is over *all* ways of splitting the indices on  $X, Y$  into two ordered subsets  $\mathbf{b}$  and  $\mathbf{b}^c$ , of any size. Using (3.88), this is exactly the form we have in the double sum in (3.84). Thus, including the prefactor of  $\prod_{i=1}^{2n} \sqrt{\omega_{+-}(z_i)}$ , we can combine these sums

<sup>10</sup>The authors would like to thank Piotr Tourkine for bringing these identities to our attention. Incidentally, note the similarity of the matrix  $V$  to the CHY matrices, given that  $\text{sgn}(i - j)$  is the propagator in fermionic QM.

<sup>11</sup>This is most easily understood via the definition of the Pfaffian in terms of differential forms. Given a two-form  $X$  on  $\mathbb{R}^{2n}$ , in coordinates  $X = X_{ij} dx^i \wedge dx^j$ , we have  $*\text{Pf}(X) = \frac{1}{n!} X^n$ . Now, given a second two-form  $Y$ , we clearly have

$$*\text{Pf}(X + Y) = \frac{1}{n!} (X + Y)^n = \frac{1}{n!} X^n + \frac{1}{(n-1)!} X^{n-1} \wedge \frac{1}{1!} Y + \cdots + \frac{1}{n!} Y^n ,$$

thus generating the sum over all partitions into two ordered subsets in eq. (3.89).

into a Pfaffian with entries

$$\begin{aligned} v_i \cdot v_j \sqrt{\omega_{+-}(z_i) \omega_{+-}(z_j)} & \left[ \frac{1}{2} \operatorname{sgn}(i-j) + \frac{S_{ij} S_{+-}}{S_{\min(i,j),+} S_{\max(i,j),-}} \right] \\ & \equiv v_i \cdot v_j S_{ij} \frac{1}{2} \left( \sqrt{\frac{S_{i+} S_{j-}}{S_{i-} S_{j+}}} + \sqrt{\frac{S_{j+} S_{i-}}{S_{j-} S_{i+}}} \right), \end{aligned} \quad (3.90)$$

for the vector part, and a similar Pfaffian for the axial part. Altogether, we find that eq. (3.84) evaluates to

$$\begin{aligned} & \frac{\operatorname{tr}(1)}{2} \operatorname{Pf} \left[ v_i \cdot v_j \frac{S_{ij}}{2} \left( \sqrt{\frac{S_{i+} S_{j-}}{S_{i-} S_{j+}}} + \sqrt{\frac{S_{j+} S_{i-}}{S_{j-} S_{i+}}} \right) \right] \\ & + \frac{\operatorname{tr}(1)}{2 \cdot 9!!} \int d^d \Psi_0 \operatorname{Pf} \left[ v_i \cdot v_j \frac{S_{ij}}{2} \left( \sqrt{\frac{S_{i+} S_{j-}}{S_{i-} S_{j+}}} + \sqrt{\frac{S_{j+} S_{i-}}{S_{j-} S_{i+}}} \right) + v_i \cdot \Psi(z_i) v_j \cdot \Psi(z_j) \right] \end{aligned} \quad (3.91)$$

where we defined  $\Psi(z) = \sqrt{\omega_{+-}(z)} \Psi_0$ . We recognise  $\Psi(z)$  as the  $q \rightarrow 0$  limit of the fermionic zero mode of the field  $\psi(z)$  on the torus in the odd spin structure.

There is also a ghost contribution in the Ramond sector, which is fortunately much more straightforward. We have

$$\left\langle e^{-\phi(z_+)/2} e^{-3\phi(z_-)/2} \right\rangle_{\beta\gamma} = S_{+-}^{3/4}, \quad (3.92)$$

which combines with the factor of  $S_{+-}^{5/4}$  from the spin field correlator. Notice in particular that the final answer is manifestly symmetric under exchange of  $z_+$  and  $z_-$ , so that it does not matter which ‘end’ of the gluing operator we write in picture  $-3/2$ .

So far our calculation has been for generic insertions  $v_i \cdot \psi(z_i)$  in eq. (3.84). To compare to the 1-loop answer of [18, 19] we specialise to the case of  $n$  external NS vertex operators of picture number 0. We find that the entries of the vector and axial Pfaffians in (3.91) can be expressed in terms of the  $\mathcal{O}(q^0)$  limits of the genus-one Pfaffians of [26] in spin structures 2 and 1, respectively. Thus we finally obtain the contribution to the one-loop CHY integrand in the Ramond sector:

$$- \frac{S_{+-}^2}{\ell^2} \left( 8 \operatorname{Pf}(M_2)|_{q^0} + \frac{8}{9!!} \operatorname{Pf}(M_1)|_{q^0} \right). \quad (3.93)$$

The first term is in precise agreement with the results of [18, 19] for spin structure 2 part of the Ramond sector contribution. This spin structure is all that is needed for the amplitude where the external kinematics are chosen to lie in fewer than eight dimensions, which is the case considered by [18, 19]. The final term in (3.93) can easily

be shown to follow from the  $q \rightarrow 0$  limit of the odd spin structure integrand in [26], and contributes when the external kinematics are in generic dimensions.

To summarise, we have computed the worldsheet correlator of our proposed gluing operator (3.77) in both the NS and R sectors. In the single trace colour structure with colour order  $\alpha$ , we obtain

$$\begin{aligned} & \int \left\langle \Delta_{\text{YM}}(z_+, z_-) \prod_{i=1}^n c(z_i) \tilde{c}(z_i) t_a^{(i)} j^a(z_i) \delta(\gamma(z_i)) \varepsilon_i \cdot \psi(z_i) e^{ip_i \cdot X(z_i)} \right\rangle_{\alpha} \\ &= \delta^{10} \left( \sum_{i=1}^n p_i \right) \int \frac{d^{10} \ell}{\ell^2} \sum J^{-1} \text{PT}(\alpha \cup \{z_+, z_-\}) (\mathcal{I}_{\text{NS}} + \mathcal{I}_{\text{R}}) \end{aligned} \quad (3.94)$$

where the integral on the left is taken over the full moduli space of the theory – including the moduli of the bosonic and fermionic gauge fields  $e$  and  $\chi$ , as well as the locations of the punctures. On the right,  $J$  is the Jacobian (3.32) from solving the 1-loop scattering equations, PT is the Parke-Taylor factor evaluated on these solutions, and

$$\begin{aligned} \mathcal{I}_{\text{NS}} &= S_{+-}^2 \left[ 8 \text{Pf}(M_3)|_{q^0} + \text{Pf}(M_3)_{\sqrt{q}} \right] \\ \mathcal{I}_{\text{R}} &= -S_{+-}^2 \left[ 8 \text{Pf}(M_2)|_{q^0} + \frac{8}{9!!} \text{Pf}(M_1)|_{q^0} \right] \end{aligned} \quad (3.95)$$

Finally, the sum inside the integral is taken both over all solutions of the 1-loop scattering equations (3.28) and also over the location of the insertion of the pair  $\{z_+, z_-\}$  jointly into the worldsheet Parke-Taylor factor. This expression is in perfect agreement with the form of the 1-loop SYM integrand found in [18, 19], extended to be valid with generic external kinematics.

### 3.4 The Gluing Operator for Gravity

We turn finally to the gluing operator for the ambitwistor string description of gravity. Unlike the models of the previous section, which suffer from various anomalies and have unwanted extra states in their spectrum, the action (1.33) defines a consistent worldsheet theory describing Type II supergravity in  $d = 10$  [13, 26]. Thus, in this case, we will evaluate the full worldsheet correlator without needing to neglect any terms ‘by hand’.

Having seen the form of the 1-loop gluing operator in SYM, its form in supergravity is not difficult to guess. We have

$$\begin{aligned} \Delta_g(z_+, z_-) = & \int \frac{d^d \ell}{\ell^2} c(z_+) \tilde{c}(z_+) c(z_-) \tilde{c}(z_-) W_{\text{GR}}(z_+, z_-) \\ & \times \left( \Delta_{\text{NS}}(z_+, z_-) + \Delta_{\text{R}}(z_+, z_-) \right) \left( \tilde{\Delta}_{\text{NS}}(z_+, z_-) + \tilde{\Delta}_{\text{R}}(z_+, z_-) \right). \end{aligned} \quad (3.96)$$

Here,  $\Delta_{\text{NS}}$  and  $\Delta_{\text{R}}$  are given by (3.53) and (3.77) respectively, as in SYM.  $\tilde{\Delta}_{\text{NS}}$  and  $\tilde{\Delta}_{\text{R}}$  are given by exactly similar operators, but constructed from the tilded fermion system that in gravity replaces the currents  $j^a$ . Again, we remark that the chirality of the spinor  $\tilde{\zeta}$  in  $\tilde{\Delta}_{\text{R}}$  should be chosen opposite / the same as that of  $\zeta$  in  $\Delta_{\text{R}}$  to obtain the loop integrand in Type IIA / IIB supergravity. Finally, non-local part of the the gravitational gluing operator is

$$\begin{aligned} W_{\text{GR}}(z_+, z_-) = & \exp \left( \frac{\ell^2}{2} \int_{\Sigma} e(x) \omega_{+-}^2(x) \right) \\ & \times \exp \left( \frac{\ell^2}{2} \int_{\Sigma \times \Sigma} \chi(x) \omega_{+-}(x) S(x, y) \chi(y) \omega_{+-}(y) \right) \\ & \times \exp \left( \frac{\ell^2}{2} \int_{\Sigma \times \Sigma} \tilde{\chi}(x) \omega_{+-}(x) S(x, y) \tilde{\chi}(y) \omega_{+-}(y) \right) \end{aligned} \quad (3.97)$$

and depends on both the fermionic gauge fields  $\chi, \tilde{\chi}$ . As above, this reflects the fact that since the gluing operator modifies the bosonic scattering equations, so too must it modify both fermionic symmetries.

As in SYM, individually the NS and R parts of the gluing operator fail to be BRST invariant, with the failure localised on the boundary of moduli space where  $z_+ = z_-$ , but this failure cancels between the two sectors. Again this can be understood as a consequence of target space supersymmetry, which is not manifest in our RNS description.

The path integrals over the  $\psi$  and  $\tilde{\psi}$  systems (and the associated ghosts) can be performed independently, and give contributions identical to the corresponding terms in SYM. Altogether we obtain that the correlation function of the gravitational gluing operator and  $n$  external NS sector particles (describing either gravitons, dilatons or

the  $B$ -field) gives

$$\begin{aligned} & \int \left\langle \Delta_g(z_+, z_-) \prod_{i=1}^n c(z_i) \tilde{c}(z_i) \varepsilon_i \cdot \psi(z_i) \tilde{\varepsilon}_i \cdot \tilde{\psi}(z_i) e^{ip_i \cdot X(z_i)} \right\rangle \\ &= \delta^{10} \left( \sum_{i=1}^n p_i \right) \int \frac{d^{10} \ell}{\ell^2} \sum J^{-1} (\mathcal{I}_{\text{NS}} + \mathcal{I}_{\text{R}}) (\tilde{\mathcal{I}}_{\text{NS}} + \tilde{\mathcal{I}}_{\text{R}}) . \end{aligned} \quad (3.98)$$

Here  $J$  is again the Jacobian from solving the one-loop scattering equations,  $\mathcal{I}_{\text{NS}}$  and  $\mathcal{I}_{\text{R}}$  were given in (3.95), and  $\tilde{\mathcal{I}}_{\text{NS}}$ ,  $\tilde{\mathcal{I}}_{\text{R}}$  are exactly analogous but with the tilded polarisation vectors. As always, the sum is over all solutions to the 1-loop scattering equations. In [18, 19] it was shown that, this final expression is equivalent to both the 1-loop integrand of supergravity, and the genus 1 ambitwistor string calculation of [26]. Here we have derived it working purely with correlation functions in a CFT at genus zero.

### 3.5 Discussion

In this section, we have defined the gluing operator for the ambitwistor string, which provides a world-sheet CFT description of known recursive definitions of scattering amplitudes in the CHY framework. It provides a systematic way of understanding ‘off-shell’ scattering equations in the context of the BRST closure of the gluing operator.

It seems certain that the gluing operator we construct is nothing other than the string propagator, computed according to the general method of [159, 160], but specialised to the case of the ambitwistor string worldsheet action. Thus, perhaps the most pressing question arising from this work is to find an *ab initio* derivation of the gluing operator as the ambitwistor string propagator. Part of the attraction of the operator approach to superstrings was that it allowed one easier access to higher genus worldsheet correlators. Hopefully, understanding the ambitwistor string propagator should similarly allow one to use the gluing operator to construct multi-loop integrands. Indeed, the insertion of two gluing operators does reproduce the structures found by [20], but some questions regarding BRST invariance remain open.

We note that the operator approach to ambitwistor strings has begun to be explored in [161, 162]; it would be interesting to relate that work to the ideas here.





# Chapter 4

## Open Questions and Future Research

We have studied generalisations of the twistor- and ambitwistor string in various directions. At tree level, we have described new ambitwistor string theories that compute the scattering amplitudes of DBI, galileon, nonlinear  $\sigma$  model, among others.

We also addressed the question of the reduction of the ambitwistor correlators to twistor correlators in  $4d$ . The insights gained from proving the mechanism behind the reduction, chiral splitting of fermion correlators, were then used to derive a previously unknown twistor string formula for EYM.

At loop level, we constructed the ambitwistor gluing operator, which can be used to write correlators in terms of products of correlators of lower number of punctures or lower genus. This operator provides the worldsheet interpretation of several previously known recursive formulas, but also yields many new ones.

We have also initiated the study of the ambitwistor string on a group manifold, specifically  $AdS_3 \times S^3$ . The prospect of ‘ $AdS$  scattering equations’ was investigated, and the importance of Mellin space was demonstrated. We used two toy models to give elucidate the possible mechanism behind localisation, which gave rise to the first  $AdS$  amplitude given in terms of  $AdS$  scattering equations.

Many important questions are still unresolved. The central imperatives in the quest towards a more comprehensive understanding of the ambitwistor string remain to study higher loop order amplitudes, scattering on curved backgrounds, the relationship to string theory proper, and fully non-perturbative aspects.

## Ambitwistor Strings on $AdS$

Possibly the most exciting avenue for future research is the ambitwistor string on a group on  $AdS$ . The prospect of a closed form expression for  $n$ -point scattering on  $AdS$  is tantalising, even if the formula will necessarily involve more complex structures.

While our computational control over the gravity model is not yet sufficient to address the question of localisation in the moduli space, we can already speculate about the possible outcomes.

The best case scenario is exemplified by the bi-adjoint scalar toy model we briefly alluded to in the text. We found that the signature of localisation, the exponential dependence on the moduli, arises when making use of the Mellin space expression for  $AdS$  integrals. A more convenient way, albeit slightly imprecise, of saying this is that the operator  $H = j^2$  becomes diagonal in Mellin space. As a result, the correlators in this model localise *in Mellin space*, and we find a scattering equations based formula for the *Mellin amplitude*. As explained in the text, Mellin space is a supremely natural arena for studying CFT correlators, since the Mellin amplitude is rational and has a simple flat space limit.

Explicit computation of the four-point function of holomorphic stress tensors suggests that the situation is not that simple, and even after going to Mellin space the operator  $H$  does not become diagonal. We are in the process of investigating if the ‘generalised Mellin space’ of [143], developed for Witten diagrams with spin, can be used to diagonalise  $H$ . If this succeeds, it would immediately lead to a scattering equation based formula for the gravity Mellin amplitude.

In the absence of the necessary computational control over the gravity model, we are currently investigating the heterotic toy model, which essentially describes Yang-Mills coupled to an exotic gravity theory.

The correlator of  $n$  holomorphic currents can be computed in closed form, and one can explicitly check that even in Mellin space the operator  $H$  acting on this correlator is not diagonal. It is however already considerably simpler than in position space, and it is conceivable that with the right tool the action of  $H$  can be diagonalised. Explicitly,  $H$  turns into a ‘number-valued’ (as opposed to ‘derivative-valued’) matrix acting on a certain space of chiral conformal blocks. A better understanding of this matrix might lead to a new kind scattering equations based formula for gluon scattering on  $AdS$ , and provide valuable clues on how to proceed in the gravity model.

Another possibility is that there may simply be no way to diagonalise  $H$  in the gravity model, and therefore the moduli space integral does not localise to the solutions

of some equations. If so it would be interesting to understand the reasons for this, and how diagonalisability emerges in the flat space limit. Addressing these questions will likely involve the use of the Morse cycles described in [14].

Moreover, even if  $H$  does not diagonalise, it might be interesting to study the representation of the correlator *before* doing the moduli space integral. With the correct prescription for the integration cycles, this still ought to be a valid formula for scattering in  $AdS$ . Constructing a proof of this formula would require new technology and may inform a deeper understanding of ambitwistor strings on more generic backgrounds.

The natural next step on the path to other backgrounds is to choose the background supported by RR flux, such that  $AdS_3 \times S^3$  can be described in terms of the supergroup  $PSU(1,1|2)$ . Constructing a string theory on this group requires the Green-Schwarz formalism [98–100] or pure spinors, and ambitwistor techniques along the lines of [149] can be used to construct a tractable ambitwistor model.

This might open up the way towards  $AdS_5 \times S^5$  which is (the bosonic part of) a super coset

$$AdS_5 \times S^5 \simeq \frac{PSU(2,2|4)}{SO(4,1) \times SO(5)} \Big|_{bos} ,$$

and has a much richer structure.

A complementary approach to the (super-)group approach to ambitwistor string on  $AdS$  is to take as target space the so-called *embedding space* of  $AdS$ . The physical, curved space arises as the projectivisation of flat space, which implemented by on the world-sheet by gauging a certain  $U(1)$  current. This strategy was used in [163] to study ambitwistor strings for certain target space CFTs, but using it for  $AdS_3 \times S^3$  might avoid some of the difficulties encountered there.

## Ambitwistor Strings, EYM and the Holomorphic Wilson Loop

It has proven notoriously difficult to find a consistent world-sheet model that gives rise to Einstein-Yang-Mills amplitudes. Both the CHY formula [10] in arbitrary dimensions and our formula in  $4d$  are very suggestive of a world-sheet origin, and indeed we have constructed several models which reproduce some, but never all features of these formulae. The crucial missing ingredient always seems to be the interplay between world-sheet ‘super-symmetry’ and target space super-symmetry, in the sense that the inclusion of the operator generating a gluon trace has to break half the target space supersymmetry. In conventional string theory, this occurs by introducing a

boundary of the world-sheet, and the boundary conditions naturally break the world-sheet supersymmetry in half. For several reasons it appears unnatural to introduce a boundary in the chiral ambitwistor action, even though this might be an interesting avenue in its own right [164].

A promising, but so far unsuccessful approach to computing Einstein-Yang-Mills amplitudes in the ambitwistor string [13] is the operator

$$\log \det \bar{D} = \log \det(\bar{\partial} + \bar{\mathcal{A}}) , \quad (4.1)$$

where the  $\bar{D}$  is the covariant derivative  $\bar{D} = \bar{\partial} + \bar{\mathcal{A}}$  in twistor space pulled back to the worldsheet. This operator has appeared before, in the twistor action for  $\mathcal{N} = 4$  SYM, and was discussed e.g. in [165, 155, 166]. Most importantly, it is not target-space gauge invariant, because the determinant of the  $\bar{D}$  operator is anomalous and transforms according to Quillen as

$$\det \bar{D} \longrightarrow \exp \left( \int_{\Sigma} g^{-1} \partial g \wedge \bar{\mathcal{A}} \right) \cdot \det \bar{D} , \quad \text{as } \bar{D} \rightarrow g \bar{D} g^{-1} , \quad (4.2)$$

under gauge transformations.

The form of the known formulae suggest that for each trace one such  $\log \det \bar{D}$  operator, expanded perturbatively to the appropriate order, is to be inserted onto the worldsheet. However, both worldsheet and target-space gauge invariance dictate some extra structure of the insertions. The most generic insertion seems to be

$$\int d^2 \chi \log \det(\bar{\partial} + \bar{\mathcal{A}}(X + \theta \tilde{\psi})) . \quad (4.3)$$

This expression needs some explanation. Firstly, the gauge field  $\bar{\mathcal{A}}$  is of course a function of the phase space coordinates. On the worldsheet, for a momentum eigenstate, we have

$$\bar{\mathcal{A}}(X) = \bar{\delta}(k \cdot P) (P^\mu + \psi^\mu k \cdot \psi) \epsilon_\mu e^{i k \cdot X} , \quad (4.4)$$

where  $A_\mu = \epsilon_\mu e^{i k \cdot X}$  are the components of the target space connection one-form pulled back to  $\Sigma$  via  $X$ . We shift its argument by a fermion bilinear  $\theta \tilde{\psi}$ , where  $\tilde{\psi}$  is the familiar worldsheet field, but  $\theta$  is a non-dynamical, holomorphic section of  $\Pi T^{1/2}$ . Picking a basis  $\gamma_a(\sigma)$  for  $T^{1/2}$ , we have  $\theta = \gamma_1 \theta_1 + \gamma_2 \theta_2$ . The integral is over the moduli space of such sections, which is two dimensional ( $d^2 \theta = d\theta_1 d\theta_2$ ) and can easily be performed by

hand, resulting in

$$\int \text{PT} \sum_{i < j \in T} k_i \cdot \tilde{\psi}_i \bar{\mathcal{A}}_i k_j \cdot \tilde{\psi}_j \bar{\mathcal{A}}_j \frac{z_i - z_j}{\sqrt{dz_i dz_j}} \prod_{\substack{k \in T \\ k \neq i, j}} \bar{\mathcal{A}}_k . \quad (4.5)$$

This is precisely the desired expression for a trace  $T \in S_n/\mathbb{Z}_n \times \mathbb{Z}_2$  of gluons, in the “integrated” version. Note that it is target-space gauge invariant, as the additive anomaly contribution eq. (4.2) does not survive the fermionic integral. The main caveat to this operator is that we do not fully understand its behaviour under the part of the BRST pertaining to the  $P\tilde{\psi}$  constraint. In particular it behaves as

$$\begin{aligned} \exp\left(\rho \oint \tilde{\gamma} P\tilde{\psi}\right) &\cdot \int d^2\theta \log \det(\bar{\partial} + \bar{\mathcal{A}}(X + \theta \tilde{\psi})) \\ &= \int d^2\theta \log \det(\bar{\partial} + \bar{\mathcal{A}}(X + (\theta + \rho \tilde{\gamma}) \tilde{\psi})) , \end{aligned} \quad (4.6)$$

under a finite transformation with the fermionic parameter  $\rho$  and using the scattering equations. Notice that when  $\tilde{\gamma}$  is on-shell, i.e. holomorphic, we can absorb the BRST transformation by a change of integration variable which shows that the proposed operator is indeed BRST invariant. This argument does however not work off-shell, where  $\bar{\partial}\tilde{\gamma} \neq 0$ . It nevertheless reproduces all amplitudes correctly in gauge where all PCOs are coincident vertex operators. Finding an extension of this to when the PCOs are at arbitrary locations is either the key to constructing a model for EYM, or impossible.

The currently most promising interpretation will likely require to take a new perspective on the worldsheet geometry of the original model [13]. The rough idea is that the  $\log \det \bar{D}$  operator is associated to a  $\mathbb{CP}^{1|0}$  subvariety  $\mathcal{R}$  of the worldsheet, and the section  $\theta$  is the modulus of the embedding of  $\mathcal{R}$  into the worldsheet (after super-conformal invariance of both the subvariety and the worldsheet are used to fix as many of the embedding-moduli as possible). Since the connection in  $\bar{D}$  is pulled back along the embedding, this picture explains why its argument is  $X + \theta \tilde{\psi}$  in eq. (4.3). Now it becomes natural to integrate over all possible ways to embed  $\mathcal{R}$  into the worldsheet. The restriction to a holomorphic modulus  $\theta$  suggests that the target of the embedding is a  $\mathbb{CP}^{1|1}$  cs-manifold [167, 168].

The starting point would be to adjoin to the worldsheet coordinates  $z, \bar{z}$  the fermionic coordinate  $\theta$ , and define the superfields

$$\mathcal{X} = X + \theta \psi , \quad \Psi = \bar{\psi} + \theta P \quad (4.7)$$

in terms of the familiar ambitwistor fields  $X, P, \psi, \bar{\psi}$ . The action in these coordinates reads

$$S = \int d\theta \Psi_\mu \bar{\partial} \mathcal{X}^\mu + \frac{\eta^{\mu\nu}}{2} \Psi_\mu \xi \Psi_\nu \quad (4.8)$$

with the super-connection  $\xi = \tilde{e} + \chi \partial_\theta$ . The second term gives rise to the gauge constraints  $H = \frac{1}{2}P^2$  and  $\bar{G} = \bar{\psi}P$ , while the remaining gauge constraint  $G = \psi \cdot P$  actually arises as the geometric super-derivative  $\partial_\theta$ . Preliminary explorations suggest that this reformulation may be the right way to understand the EYM correlators. The proper understanding of this model is not yet complete and is work in progress – we shall only present some circumstantial evidence in support of this approach.

The CHY formula suggests there must also be “fixed” versions of the trace operator (4.5), with respect to the  $T, P^2$  and  $P\tilde{\psi}$  constraints. Indeed, in the -2 picture<sup>1</sup> of the  $\tilde{\gamma}$  constraint, the operator can be written

$$\int \text{PT} \delta(\tilde{\gamma}_i) \bar{\mathcal{A}}_i \delta(\tilde{\gamma}_j) \bar{\mathcal{A}}_j \frac{z_{ij}}{\sqrt{dz_i dz_j}} \prod_{\substack{k \in T \\ k \neq i, j}} \bar{\mathcal{A}}_k . \quad (4.9)$$

It seems plausible that descent along the  $\tilde{\gamma}$  direction gives rise to eq. (4.5), but this can only be made precise after understanding the interplay between this operator and the  $P\tilde{\psi}$  constraint. One possibility of writing eq. (4.9) in terms of the fundamental operator is

$$\int d^2\theta_a \left( \int \delta(\tilde{\gamma}) \theta \bar{\mathcal{A}} \frac{\delta}{\delta \bar{\mathcal{A}}} \right)^2 \log \det(\bar{\partial} + \bar{\mathcal{A}}(X + \theta \tilde{\psi})) , \quad (4.10)$$

which also manifests target-space gauge invariance. Note that we could drop the  $\theta\tilde{\psi}$  from the argument of the connection. The operator in the -1 picture might then be

$$\int d^2\theta_a \left( \int \delta(\tilde{\gamma}) \theta \bar{\mathcal{A}} \frac{\delta}{\delta \bar{\mathcal{A}}} \right) \log \det(\bar{\partial} + \bar{\mathcal{A}}(X + \theta \tilde{\psi})) . \quad (4.11)$$

However, we have not yet found a natural interpretation in terms of the embedding picture that explains the origin of the functional variation.

Another way to write the fixed, picture -2, operator is

$$\left( \int \bar{\mathcal{A}} \frac{\delta}{\delta \bar{\mathcal{A}}} \right)^2 \log \det(\bar{\partial} + \bar{\mathcal{A}}(X + \theta \tilde{\psi})) , \quad (4.12)$$

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<sup>1</sup>We view the whole operator to saturate the zero modes, hence we call this the -2 picture.

without an integral over the embedding modulus. This formulation avoids the target-space gauge anomaly eq. (4.2) by taking two variations at different worldsheet points. Since the anomaly acts by adding the integral of a local expression, it will be annihilated by these two derivatives.

In order to justly call this operator “fixed,” it should, of course, saturate the  $\tilde{\gamma}$  zero modes. This can be achieved by first interpreting the  $\tilde{\gamma}$  transformations to act geometrically and shift the embedding. In other words,  $\oint \tilde{\gamma} P \tilde{\psi}$  should also act on the modulus  $\theta$ . Then the zero modes are saturated by requiring the  $\tilde{\gamma}$  transformation to be zero at the locus of the operator, i.e. the entire embedded  $\mathbb{CP}^{1|0}$ . As the zero modes are holomorphic, this forces them to vanish everywhere. At the same time it explains why the operator eq. (4.5) is invariant under  $\tilde{\gamma}$  transformations; since it is integrated over the whole moduli space of the embedding, shifting the modulus under the integral can be absorbed into a change of coordinates on the moduli space. This kind of reasoning is reminiscent of the D-brane boundary condition for worldsheet SUSY in type II string theory. It is not clear, however, how far this analogy can be stretched, and if these arguments can eventually be made precise.

Note that, if the above arguments turn out to be true, the final amplitude will not depend on which particular embedding modulus  $\theta$  is chosen for the fixed operator. This is again because a shift in the modulus can be undone by a BRST transformation, which can be “partially integrated” on moduli space to hit all other vertex operator insertions, which will be BRST closed.

For completeness we mention also the version that is fixed with respect to the  $c\tilde{c}$  constraints,

$$\left( c\tilde{c} j \frac{\delta}{\delta \bar{\mathcal{A}}} \right)^3 \log \det \bar{D} \quad (4.13)$$

where  $j$  was defined in eq. (4.4). It is gauge invariant by the same reasoning as above, but also still awaits a complete interpretation in terms of the worldsheet geometry.

## Towards the non-linear Field Equations

It is worth mentioning that all the above operators (after performing the moduli integral or functional variation) can be expressed in terms of the *holomorphic Wilson loop* operator [155, 166]. In this context it is an operator on<sup>2</sup>  $\mathbb{CP}^1$ , defined as the

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<sup>2</sup>Depending on how we will eventually view the worldsheet geometry, it may be crucial that the connection is pulled back to a  $\mathbb{CP}^1$ , where the obstruction to finding the holomorphic Wilson loop vanishes identically. It is conceivable that the original,  $\mathcal{N} = 2$ , worldsheet does not admit this Wilson

solution to

$$\bar{\partial}U(\sigma, \sigma_0) = \bar{\mathcal{A}}U(\sigma, \sigma_0) , \quad \text{with } U(\sigma_0, \sigma_0) = \mathbb{1} . \quad (4.14)$$

In analogy to the real Wilson line, we may write a solution formally by using the “path-ordered exponential”, as

$$U(\sigma, \sigma_0) = \mathcal{P} \exp - \int \omega_{\sigma_0}^\sigma \wedge \bar{\mathcal{A}} , \quad (4.15)$$

which essentially gives rise to an expression like eq. (4.5), bar some minor factors carrying conformal weight, which may be attributed to the embedding modulus. Quite interestingly, whenever the start and end point of the Wilson line coincide, i.e. it becomes a Wilson *loop*, the operator necessarily becomes trivial, as the  $\mathcal{F}^{0,2}$  part of the curvature identically vanishes on the 1-dimensional worldsheet. Hence, in order to define a non-trivial, gauge invariant operator, one needs to have at least two insertions into the Wilson line, before closing it onto itself. These insertions are, just as in the real Wilson line, nothing but the functional variations with respect to the connection.

Putting aside the open question regarding off-shell BRST closure, we may ask how the full, non-linear equations of motion for EYM might arise from the worldsheet model and the operator insertion. In the original type II ambitwistor model the supergravity equations of motion arise as quantum anomalies in the BRST charges [47]. In particular, the worldsheet dynamics remains free, but the BRST currents  $\mathcal{G}, \bar{\mathcal{G}}, \mathcal{H}$  are modified to include the background fields. The background EOMs then appear as the coefficients of higher order poles in the OPEs of these BRST currents. It is however not clear how to incorporate a background Yang-Mills field into the present BRST charges, although important progress has been made [17, 16]. Lacking definitive knowledge of the full BRST charge we may nevertheless attempt to understand from which worldsheet structure the non-linear background equation of motions for the Yang-Mills field could arise.

We direct our attention to the part  $\gamma\psi P$  of the BRST charge which makes sense in the flat background setting. Consider including a background field in the definition of the operator eq. (4.3) and expanding it to some order in the quantum fluctuations. The result will be a colour-trace containing segments like

$$\cdots U_B(\cdot, \sigma_i) \ c\tilde{c}\delta(\gamma_i) \psi_i^\mu A_{i\mu} U_B(\sigma_i, \cdot) \cdots \quad (4.16)$$

---

loop. In order to accommodate gluons, the worldsheet geometry would then have to be changed, which reduces to  $\mathcal{N} = 1$  and removes the obstruction.



where the vertex operator is in the fixed picture,  $A_\mu(\sigma_i)$  is a ‘quantum fluctuation’ of the target space connection one-form and  $U_B$  is the holomorphic Wilson line eq. (4.15) of the target space background connection with an appropriate extension of eq. (4.4). As a fluctuation,  $A_\mu(\sigma_i)$  transforms in the adjoint of the gauge group and eq. (4.16) is hence background gauge invariant. Now assume that we can find a modification of the present operators so as to realise the following OPE

$$P_\mu(z) U(0, \cdot) \sim -\frac{1}{z} A_\mu^B(X(z)) U(0, \cdot) \quad (4.17)$$

where  $A^B$  is the background connection. This can be interpreted as a translation of the defining equation for a Wilson line eq. (4.14) into field-space. Notice that since  $A^B$  transforms as a connection, this OPE is again target-space gauge invariant. Using this OPE it is not hard to show that double contractions of eq. (4.16) with the BRST current  $j$  provide the necessary structures for the full, non-linear field equations to arise. As example one may show that the fluctuation  $A$  must satisfy the Yang-Mills equation of motion given by the background field  $A^B$  in order to avoid quantum inconsistencies in the form of higher order poles with the BRST currents on the worldsheet.

It would be interesting to understand the relationship of this perspective to the results of [16], where the Yang-Mills equations arise from a modification of the worldsheet gauge algebra to include the background gauge field

$$G = (P_\mu + A_\mu^a j_a) \psi^\mu, \quad H = (P + A^a j_a)^2 + [A_\mu, A_\nu]^a j_a \psi^\mu \psi^\nu \quad (4.18)$$

in the heterotic ambitwistor model, and examining the consistency conditions arising through quantization.

## YM from the Holomorphic Log Det

It is a fascinating question whether, or how much of, the techniques that do miracles at tree level continue to work at loop level or even non-perturbatively.

One idea for studying this, still in its infancy, builds on the connected prescription twistor string formula of 4d YM. The tree level amplitudes are known to arise as the expectation value of the operator

$$\int d\mu_d \log \det (\bar{\partial} + \bar{\mathcal{A}}) \Big|_c \quad (4.19)$$

in holomorphic supersymmetric Chern-Simons theory

$$S_{\text{CS}}[\mathcal{A}] = \int_{\mathbb{CP}^{3|4}} D^{3|4} Z \operatorname{tr} \left( \mathcal{A} \bar{\partial} \mathcal{A} + \frac{2}{3} \mathcal{A}^3 \right) \quad (4.20)$$

on twistor space. Here  $\mathcal{C}$  is a holomorphic curve of degree  $d$  and  $d\mu_d$  is the measure on the moduli space of such curves. This expression is known to generate the tree level amplitudes for  $n$  particles by expanding the logarithm and taking the  $n^{\text{th}}$  term in the expansion

$$\int d\mu_d \prod_{i=1}^n \frac{dz_i}{z_i - z_{i+1}} \bar{\mathcal{A}}_i(Z(\sigma_i)) = \int d\mu_d \text{PT} \prod_{i=1}^n \bar{\mathcal{A}}_i(Z(\sigma_i)) \quad (4.21)$$

Given that the value of the action evaluated on a classical action is zero, this happens to also be the leading term in the expansion around the classical limit. Taking this perspective, it becomes conceivable that

$$\left\langle \int d\mu_d \log \det (\bar{\partial} + \bar{\mathcal{A}}) \Big|_{\mathcal{C}} \right\rangle \quad (4.22)$$

is in fact a generating functional for all loop amplitudes. A very exiting question to address in this framework is the status of the ‘refined scattering equations’ for the loop integrand [58].

This claim survives several simple checks, and we are working on a more thorough investigation. This is inspired by recent work [169–172, 18, 15, 58] on the relation between tree-level and loop-level amplitudes in general supersymmetric theories and the ambitwistor context.

## Scattering Equations from Equivariant Localisation

The localisation of the moduli space integral in ambitwistor strings has been studied from many different perspectives. They all have, however, the common starting point of diagonalising the operator  $H$ . This is natural in flat space, where plane waves are available, but the study of ambitwistor strings on  $AdS$  suggests that it may not always be possible to diagonalise  $H$ . Therefore a different framework for understanding the localisation could be necessary.

One such alternative approach currently under development by the authors is to use equivariant localisation with respect to the group action of the currents  $(T, H)$  induced on  $T^*\mathcal{M}$ . This makes use of the isomorphism  $K_{\Sigma}^2[n] \simeq T^*\mathcal{M}_{0,n}$ , which means  $(T, H)$  can be seen as one-forms on the cotangent bundle to the moduli space  $\mathcal{M}_{0,n}$ . This can

be dualized to a vector field on the space  $T^*\mathcal{M}_{0,n}$  using the canonical symplectic form on the cotangent bundle  $T^*(T^*\mathcal{M}_{0,n})$ . In the flat ambitwistor string this is computed straightforwardly, and the components of the vector field in the base manifold are essentially

$$\sum_{j \neq i} \frac{p_i \cdot p_j}{z_i - z_j} \frac{\partial}{\partial z_i}.$$

The fixed points of this vector field are precisely the solutions to the scattering equations. (The components of the vector field along the fibres do not carry any important information in the flat space model.)

In flat space, this approach is of course entirely equivalent to other known approaches, and is not particularly exciting, because the group, with respect to which the equivariant localisation happens, is abelian. It may, however, be a more appropriate starting point for the generalisation to  $AdS$ .

As briefly mentioned above, in the heterotic  $AdS$  toy model, the operator  $H$  acts like a matrix in a certain space of conformal blocks. We do not have a good handle on this matrix yet, but preliminary computations suggest it might have the right properties to facilitate equivariant localisation of the correlator with respect to some appropriate non-abelian group.



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# Appendix A

## Correlators for $S_{YM,\Psi}$

Here we give the proof of theorem 1. In the main text several versions of the present idea are realised. We will demonstrate and prove the mechanism in the simplest setting, which already contains all necessary ingredients, and comment on adaptations and restrictions afterwards. Concretely we use a single free fermion  $\rho^a$  and a generic level zero current  $j^a$ . The fields have the same OPEs as above, that is  $j^a$  form a current algebra and  $\rho^a$  are in the adjoint presentation of the  $j$ -algebra, i.e.

$$\rho^a(\sigma)\rho^b(0) \sim \frac{1}{\sigma}\delta^{ab} , \quad j^a(\sigma)j^b(0) \sim \frac{1}{\sigma}f^{abc}j^c , \quad j^a(\sigma)\rho^b(0) \sim \frac{1}{\sigma}f^{abc}\rho^c . \quad (\text{A.1})$$

The strategy of the proof is as follows: both the full space-time amplitude  $\mathcal{A}(g, h)$  and the world-sheet correlator  $A(g, h)$  are a (multiple) sum of simple terms. The sum in  $\mathcal{A}$  is over trace sectors as well as a choice of gluon labels, while the sum in  $A(g, h)$  is simply the Wick expansion of the expectation value. Schematically we get

$$\mathcal{A} = \sum_{x \in X} \mathcal{A}(x) \quad \text{and} \quad A = \sum_{y \in Y} A(y) \quad (\text{A.2})$$

where  $X, Y$  are sets labelling the trace sectors and organisation of sets of Wick contractions respectively. Then we will show that  $x \in X \Rightarrow x \in Y$  and  $y \in Y \Rightarrow y \in X$ . Upon showing that each element in  $X, Y$  is unique we get  $X = Y$ . Along the way we will see that  $A(x) = \mathcal{A}(x)$ , hence establishing  $A = \mathcal{A}$ .

To clarify the structure of the discussion, we firstly only insert integrated vertex operators on the world-sheet – which corresponds to considering the full Pfaffian in the CHY formula – keeping in mind that to get a non-vanishing result we need to go over to the reduced Pfaffian. That step will be taken at the end.

So we will have to examine the correlation function of two types of operators,

$$\mathcal{O}^{gl} = k \cdot \Psi \, t \cdot \rho + t \cdot j \quad \text{and} \quad \mathcal{O}^{gr} = k \cdot \Psi \, \epsilon \cdot \Psi + \epsilon \cdot P , \quad (\text{A.3})$$

for (one half of) the gluon and graviton integrated vertex operators respectively. The claim is that the string-worldsheet correlator

$$A(g, h) := \left\langle \prod_{a \in g} \mathcal{O}_a^{gl} \prod_{a \in h} \mathcal{O}_a^{gr} \right\rangle \quad (\text{A.4})$$

where  $g$  and  $h$  are the sets containing the gluon and graviton labels respectively, is equal to (one part of the CHY representation of) the full space-time amplitude

$$\mathcal{A} = \sum_{\text{trace sectors}} \mathcal{C}_1 \cdots \mathcal{C}_m \, \text{Pf} \, \Pi , \quad (\text{A.5})$$

where the sum goes over all trace sectors possible. In particular, it includes a sum over the number of traces  $m = 1, \dots, \lfloor |g|/2 \rfloor$ . The matrix  $\Pi$ , defined in [10], of course depends on the trace sector.

The main step in going between the representations two is the identity eq. (1.106), which we repeat here for the readers convenience

$$\sigma_{ab} \, \mathcal{C}(T) = \mathcal{K}(b, a|T) , \quad (\text{A.6})$$

with  $\mathcal{K}$ , the ‘comb structure’ defined in the main text. Its arguments are the unordered set  $T$  and two of its elements,  $a, b \in T$ . Using the anti-symmetry and multi-linearity of the Pfaffian, expression eq. (A.5) can be brought into the form

$$\sum_{\substack{\text{trace} \\ \text{sectors}}} \sum_{\substack{a_1 < b_1 \in T_1 \\ \dots \\ a_m < b_m \in T_m}} \mathcal{K}(a_1, b_1|T_1) \cdots \mathcal{K}(a_m, b_m|T_m) \, \text{Pf} \, M(h, \{a_i\}, \{b_i\}|h) . \quad (\text{A.7})$$

This is the representation of the amplitude which the world-sheet correlator eq. (A.4) will land us on.

Let us now consider evaluating the correlator  $A(g, h)$ . We will see that it gives rise to a multiple sum over terms, which turn out to be the same that eq. (A.7) sums over. The first step is to expand the product of all the  $\mathcal{O}^{gl}$ s into a sum. The sum is over all ways of putting either a  $k\Psi\rho$  or a  $j$  at each gluon insertion. This is a binary choice so it leads to  $2^{|g|}$  terms. Name the set of gluon labels which carry a  $k\Psi\rho$  insertion by  $e$  for each term. The path integral over the  $\Psi$  field can now be performed for each

term individually. Since  $\Psi$  is fermionic, the path integral vanishes unless  $|e|$  is even. Define  $m := |e|/2$ , which is integer. The result of this path integral is of course simply a factor of

$$\text{Pf } M(h, e|h) \quad (\text{A.8})$$

for each term in the sum, by the standard reasoning described for example in [13]. note that, since  $e$  only appears once, the Pfaffian depends on the ordering of the elements in  $e$ . Now the correlator  $A$  is a sum over ways of partitioning  $g$  into  $e$  and  $g - e$ , with the condition that  $|e|$  be even, and each term in the sum looks like<sup>1</sup>

$$\langle \prod_{a \in e} \rho_a \prod_{a \in g-e} j_a \rangle \text{Pf } M(h, e|h) . \quad (\text{A.9})$$

It should be clear that the remaining worldsheet correlator will give rise the product of  $\mathcal{K}$ s and the remaining sum. Let us see how this happens in detail. Performing the Wick expansion of the  $\rho, j$  correlator breaks it down into a product of smaller pieces, so far until each factor contains precisely two (i.e. a pair of)  $\rho$  insertions accompanied by some subset of the  $j$  insertions. Label the pair of  $\rho$  insertions into the  $i^{\text{th}}$  factor by  $a_i, b_i$  and the accompanying set of  $j$  insertions by  $T_i$ . Wick expansion makes sure that each choice of pairs and each choice of accompanying  $j$  insertions appears at least once and only once. Schematically we get

$$\langle \prod_{a \in e} \rho_a \prod_{a \in g-e} j_a \rangle = \sum_{\text{partitions}} \prod_{i=1}^m \langle \rho_{a_i} \rho_{b_i} \prod_{c_i \in T_i} j_{c_i} \rangle . \quad (\text{A.10})$$

The remaining correlator is now easily evaluated using the OPEs eq. (A.1) to give

$$\langle \rho_a \rho_b \prod_{\substack{c \in T \\ c \neq a, b}} j_c \rangle = \mathcal{K}(a, b|T) . \quad (\text{A.11})$$

Note that the symmetry properties of the function  $\mathcal{K}$  in its arguments naturally arise from the statistics of the fields  $\rho, j$ .

Actually, performing the Wick expansion in eq. (A.10) does not preserve the order of the  $\rho$  insertions, so, as they are fermions, a factor of  $(-1)$  might appear. We can absorb this factor by bringing the rows/columns of the matrix  $M$  into the same order as the  $\rho$  appear on the *rhs* of eq. (A.10). Then eq. (A.9) becomes

$$\sum \mathcal{K}(a_1, b_1|T_1) \cdots \mathcal{K}(a_m, b_m|T_m) \text{Pf } M(h, \{a_1, b_1, \dots, a_m, b_m\}|h) , \quad (\text{A.12})$$

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<sup>1</sup>From now onwards we omit the colour structure and abbreviate  $t_a \cdot \rho(\sigma_a) = \rho_a$  and  $t_a \cdot j(\sigma_a) = j_a$ .

which is precisely the summand appearing in the full space-time amplitude<sup>2</sup>. We repeat that Wick expansion ensures that every possible configuration of the summand is summed over, each term appearing at least once and only once.

We have shown that the expressions  $A$  and  $\mathcal{A}$  are sums over the same simple terms, involving  $\mathcal{K}$ s and the corresponding  $\text{Pf } M$ . To clarify, on the one hand, the sum in  $A$  goes over different ways of choosing  $m$  pairs  $a_i, b_i$  from  $g$  and different ways of forming  $m$  unordered sets  $T_i$  from the labels left over, as well as the sum over  $m$ . The set  $X$  mentioned above contains as elements the ways of making such choices. On the other hand, the sum in  $\mathcal{A}$  goes over ways of splitting the labels  $g$  into  $m$  unordered subsets  $T_i$  and the ways of picking a pair from each subset, as well as the sum over  $m$ . The set of these choices is  $Y$ . What remains to show is that the sums are actually the same or, equivalently, that each sum includes the other. To do so, we go back to the full expressions

$$A(g, h) = \left\langle \prod_{a \in g} \mathcal{O}_a^{gl} \prod_{a \in h} \mathcal{O}_a^{gr} \right\rangle \quad \text{and} \quad \mathcal{A}(g, h) = \sum_{\text{trace sectors}} \mathcal{C}_1 \cdots \mathcal{C}_m \text{Pf } \Pi . \quad (\text{A.13})$$

and argue that if a term appears in  $A$  it also appears in  $\mathcal{A}$  and vice versa. Additionally, we argue that each term appears at least once and only once in each expression, which will conclude the proof that they are equal.

It is clear that both sums contain the summation over  $m = 1, \dots, \lfloor |g|/2 \rfloor$  in them, which is to be understood as the number of traces. Take a contribution from  $A$  with  $|e| = 2m$ . Each term in the sum is uniquely determined by specifying  $m$  pairs  $\{a_i, b_i\}$  and  $m$  unordered sets  $T_i$ . As mentioned previously, Wick expansion guarantees that each term appears once and only once. A given term should have a partner in  $\mathcal{A}$  at  $m$  traces. Looking at the representation eq. (A.5) this is not straightforward to see, but the equivalent representation eq. (A.7) makes this readily apparent. The sum over trace sectors will include one term where the  $T_i$  in  $\mathcal{A}$  are precisely<sup>3</sup> the  $T_i$  in  $A$  whereupon the sums  $a_i, b_i \in T_i$  will contain one term in which all  $a_i, b_i$  in  $\mathcal{A}$  agree with those in  $A$ . This shows that each term in  $A$  has a partner in  $\mathcal{A}$ , establishing the statement  $y \in Y \Rightarrow y \in X$ . Of course Wick expansion guarantees the uniqueness of the terms in  $Y$ .

Conversely, one term in the summation in  $\mathcal{A}$  is uniquely specified by fixing a trace structure and picking out one term of the summations over  $a_i, b_i$ . In other words, it is specified by a collection of  $m$  sets  $T_i$  and a choice of pairs  $\{a_i, b_i\}$  for each set. To see

<sup>2</sup>In fact there will be additional sign factors from permutations the rows/columns in the Pfaffian.

<sup>3</sup>In a slight abuse of notation, what is called  $T_i$  in  $\mathcal{A}$  is actually  $T_i \cup \{a_i, b_i\}$  in  $A$ .

that any such term is also contained in  $A$  simply notice that the above data uniquely specifies a term in  $A$  via

$$\prod_{i=1}^m \langle \rho_{a_i} \rho_{b_i} \prod_{c_i \in T_i} j_{c_i} \rangle \text{Pf} M(h, \{\{a_i, b_i\}_i\} | h) . \quad (\text{A.14})$$

Hence, each term in  $\mathcal{A}$  has a partner in  $A$  and this establishes the statement  $x \in X \Rightarrow x \in Y$ . The uniqueness of each element follows by construction.

### The Reduced Pfaffian

The Pfaffian we discussed so far actually vanishes for physical systems, i.e. when momentum conservation, gauge invariance and the scattering equations hold. Hence it is replaced by the reduced Pfaffian  $\text{Pf}' \Pi$  defined in either of the following equivalent ways

$$\text{Pf}' \Pi := \text{Pf} \Pi_{i,j'} = \frac{(-)^a}{\sigma_a} \text{Pf} \Pi_{a,i} = -\frac{(-)^a}{\sigma_a} \text{Pf} \Pi_{a,j'} = \frac{(-)^{a+b}}{\sigma_{ab}} \text{Pf} \Pi_{a,b} \quad (\text{A.15})$$

where  $a, b$  label gravitons, with the restriction to not remove any row/column of the matrix  $B$ , and the  $i, j'$  label traces. On the other hand, we know that the expectation value of all integrated vertex operators will also vanish, and we have to insert precisely two fixed vertex operators. For an all graviton amplitude, this was discussed in [13]. It follows from BRST invariance that the amplitude is invariant under the choice of which vertex operators to take fixed/integrated. Hence, if there are at least two gravitons and arbitrarily many gluons, the full amplitude must be equal to the CHY formula. We will now show that the reduced Pfaffian also follows when using fixed vertex operators for two gluons or one gluon and one graviton, trying to present the following expressions in a suggestive form.

### Two Gluons Fixed

Denote the labels of the fixed gluon operators as  $c, d$ . With the reduced Pfaffian defined as

$$\text{Pf}' \Pi = \text{Pf} \Pi_{i,j'} \quad (\text{A.16})$$

there are two cases,  $j' = i$  or  $j' \neq i$ . In the first case the trace  $T_i$  is totally removed from the Pfaffian and we can write

$$\cdots \mathcal{C}_i \cdots \text{Pf} \Pi_{i,i'} = \frac{1}{(dc)} \cdots \mathcal{K}(c, d|T_i) \text{Pf} \Pi_{i,i'} , \quad (\text{A.17})$$

with the gluons  $c, d$  being members of the trace  $T_i$ . The factor  $\frac{1}{(dc)}$  fits into the interpretation of [13] as ghost field correlator. Note that there is no sum over choices of pairs in  $T_i$ , instead the comb  $\mathcal{K}$  appears with fixed start/end points, corresponding to the insertion of fixed vertex operators for the gluons  $c, d$ .

In the second case ( $j' \neq i$ ), name the traces such that  $c \in T_1$  and  $d \in T_2$ . Now each term in the expansion of the worldsheet correlator will look like (omitting all irrelevant factors)

$$\begin{aligned}
\frac{1}{\sigma_{cd}} \sum_{\substack{a \in T_1 \\ b \in T_2}} \mathcal{K}(c, a|T_1) \mathcal{K}(d, b|T_2) \text{Pf}(a, b, \dots) &= \mathcal{C}(T_1) \mathcal{C}(T_2) \sum_{\substack{a \in T_1 \\ b \in T_2}} \frac{\sigma_{ac} \sigma_{bd}}{\sigma_{cd}} \text{Pf}(a, b, \dots) \\
&= \mathcal{C}(T_1) \mathcal{C}(T_2) \sum_{a \in T_1} \frac{\sigma_{ac}}{\sigma_{cd}} \text{Pf}(a, \sum_{b \in T_2} \sigma_{bd} b, \dots) \\
&= \mathcal{C}(T_1) \mathcal{C}(T_2) \sum_{a \in T_1} \frac{\sigma_{ac}}{\sigma_{cd}} \text{Pf}(a, - \sum_{b \in T_1} \sigma_{bd} b, \dots) \\
&= \mathcal{C}(T_1) \mathcal{C}(T_2) \sum_{\substack{a \in T_1 \\ b \in T_1}} \frac{\sigma_{ac} \sigma_{db}}{\sigma_{cd}} \text{Pf}(a, b, \dots) \\
&= \mathcal{C}(T_1) \mathcal{C}(T_2) \sum_{a < b \in T_1} \frac{\sigma_{ac} \sigma_{db} - \sigma_{bc} \sigma_{da}}{\sigma_{cd}} \text{Pf}(a, b, \dots) \\
&= \mathcal{C}(T_1) \mathcal{C}(T_2) \sum_{a < b \in T_1} \sigma_{ba} \text{Pf}(a, b, \dots) \equiv \mathcal{C}(T_1) \mathcal{C}(T_2) \text{Pf} \Pi_{2,2'} .
\end{aligned} \tag{A.18}$$

Note that we had to use the scattering equations and the antisymmetry of the Pfaffian to arrive at the final result.

### One Gluon, One Graviton Fixed

The computation for fixing one gluon and one graviton vertex operator is largely analogous to the previous one. Moreover, BRST invariance guarantees that the final result will be as desired. Let us nevertheless demonstrate the necessary manipulations. Denote the fixed gluon by  $c$ , with  $c \in T_1$ , and the fixed graviton by  $m$

$$\begin{aligned}
\frac{1}{\sigma_{mc}} \sum_{a \in T_1} \mathcal{K}(a, c|T_1) \text{Pf}(a, \dots, \check{m}, \dots) &= \mathcal{C}(T_1) \sum_{a \in T_1} \frac{\sigma_{ca}}{\sigma_{mc}} \text{Pf}(a, \dots, \check{m}, \dots) \\
&= \mathcal{C}(T_1) \frac{1}{\sigma_{mc}} \text{Pf} \left( \sum_{a \in T_1} \sigma_{ca} a, \dots, \check{m}, \dots \right) \\
&= \mathcal{C}(T_1) \frac{1}{\sigma_{mc}} \text{Pf}(-\sigma_{cm} m, \dots, \check{m}, \dots) \\
&= \mathcal{C}(T_1) \text{Pf}(m, \dots, \check{m}, \dots) = \mathcal{C}(T_1) \text{Pf} \Pi_{1,1'} .
\end{aligned} \tag{A.19}$$

Again we had to make use of the scattering equations.

### Adaption and Restriction

As mentioned in the text, it seems not to be possible to find a level zero current via descent from  $\rho$  in a consistent way satisfying eq. (A.11). Hence, the main text contains an adaption of the system discussed above, using two fermions  $\rho^a, \tilde{\rho}^a$ , conjugate to each other. Via the descent,  $\rho^a$  gives rise to  $j^a$  while  $\tilde{\rho}^a$  gives rise to  $\tilde{j}^a$ . The OPEs between the currents and the fields are

$$\begin{aligned} \rho^a(z)j^b(0) &\sim \frac{1}{z}f^{abc}\rho^c, & \tilde{\rho}^a(z)j^b(0) &\sim \frac{1}{z}f^{abc}\tilde{\rho}^c, \\ \rho^a(z)\tilde{j}^b(0) &\sim \frac{1}{z}f^{abc}\tilde{\rho}^c, & \tilde{\rho}^a(z)\tilde{j}^b(0) &\sim 0. \end{aligned} \quad (\text{A.20})$$

We shall now examine the correlators of this system.

First, note that by taking the fixed vertex operators to be  $(\rho + \tilde{\rho})$ , the discussion above would carry over verbatim. There is a crucial difference however. The current appearing in the associated integrated vertex operator does not quite satisfy eq. (A.11), but instead

$$\langle (\rho_a + \tilde{\rho}_a)(\rho_b + \tilde{\rho}_b) \prod_{c \in T} (j_c + \tilde{j}_c) \rangle = |T| \mathcal{K}(a, b|T). \quad (\text{A.21})$$

So each contribution from a different trace sector will come with a different prefactor  $\prod_i^m |T_i|$ , spoiling the relative coefficient between partial amplitudes. As the prefactor depends on the given partition of particles into traces, it cannot be removed by a field rescaling. The origin of this factor can be understood by simply counting the ways in which a full comb can be generated. If we represent the fields  $\rho, \tilde{\rho}$  by  $+, -$  and the currents  $j, \tilde{j}$  by  $\pm, ++$  respectively, the possible contractions can be found by drawing all allowed charge flows as in figure A.1

Observe that each contraction must have exactly one insertion of  $\tilde{j}$  (represented by  $++$ ) or  $\tilde{\rho}$  (represented by  $+$ ), independent of the length  $n$  of the chain, while there are  $n - 1$  insertions of  $j$  or  $\rho$ . Summing over the possible positions of the tilded operator in the chain gives rise to the over-counting by  $|T|$ . Note that each contraction contributes exactly the same analytical & colour structure.

Having understood the (non-trivial) origin of the factor  $|T|$ , the remainder of the discussion, showing how to remove it, follows trivially. Denote  $v$  the vertex operator containing  $\rho$  and  $j$  and  $\tilde{v}$  the one containing  $\tilde{\rho}$  and  $\tilde{j}$ , either integrated or fixed. It is now clear that choosing to insert  $\tilde{v}$  at  $m$  of the gluon punctures and  $v$  at the others will give rise (following the general discussion above) to the complete color ordered

1	→	2	→	3	→	...	→	$n-1$	→	$n$
+	→	±	→	±	→	...	→	±	→	-
-	→	++	→	±	→	...	→	±	→	-
-	→	±	→	++	→	±	→	...	→	-
⋮			⋮		⋮					
-	→	±	→	...	→	±	→	++	→	-
-	→	±	→	±	→	...	→	±	→	+

Fig. A.1 Charge flows

partial amplitude with  $m$  traces

$$\mathcal{C}(T_1) \cdots \mathcal{C}(T_m) \text{ Pf } \Pi , \quad (\text{A.22})$$

which concludes the discussion



# Appendix B

## Degeneration limit of the torus Szegó kernel

The genus-one Szegó kernels have following expansions at small  $q = \exp(i\pi\tau)$ :

$$S_1(z_i, z_j|\tau) = S_{ij} \frac{1}{2} \left( \sqrt{\frac{S_{i+}S_{j-}}{S_{i-}S_{j+}}} + \sqrt{\frac{S_{j+}S_{i-}}{S_{j-}S_{i+}}} \right) + \mathcal{O}(\sqrt{q}) \quad (\text{B.1})$$

$$S_2(z_i, z_j|\tau) = S_{ij} \frac{1}{2} \left( \sqrt{\frac{S_{i+}S_{j-}}{S_{i-}S_{j+}}} + \sqrt{\frac{S_{j+}S_{i-}}{S_{j-}S_{i+}}} \right) + \mathcal{O}(\sqrt{q}) \quad (\text{B.2})$$

$$S_3(z_i, z_j|\tau) = S_{ij} + \sqrt{q} \frac{S_{i+}S_{-j} - S_{i-}S_{+j}}{S_{+-}} + \mathcal{O}(q) \quad (\text{B.3})$$

$$S_4(z_i, z_j|\tau) = S_{ij} - \sqrt{q} \frac{S_{i+}S_{-j} - S_{i-}S_{+j}}{S_{+-}} + \mathcal{O}(q) \quad (\text{B.4})$$

where

$$S_{ij} \equiv S(z_i, z_j) = \frac{\sqrt{dz_i dz_j}}{z_i - z_j} \quad (\text{B.5})$$

is the Szegó kernel on the sphere and  $z_{\pm}$  are the coordinates of the node. We can take into account the fermionic zero mode in spin structure 1 by modifying the propagator of fermions, e.g. at  $q = 0$  (on the nodal sphere) we find

$$\eta^{\mu\nu} S_{ij} \frac{1}{2} \left( \sqrt{\frac{S_{i+}S_{j-}}{S_{i-}S_{j+}}} + \sqrt{\frac{S_{j+}S_{i-}}{S_{j-}S_{i+}}} \right) + \sqrt{\omega_{+-}(z_i) \omega_{+-}(z_j)} \psi_0^\mu \psi_0^\nu \quad (\text{B.6})$$

where  $\psi_0^\mu$  is a set of  $d$  Grassmann numbers, over which the entire correlator is to be integrated using the standard Berezin integration rules. Here  $\omega_{+-}(z)$  is the  $q \rightarrow 0$  limit of the unique holomorphic one-form on the torus, normalised to have residues  $\pm 1$  at

$z = z_{\pm}$ , i.e.

$$\omega_{+-}(z) = \frac{dz (z_+ - z_-)}{(z - z_+)(z - z_-)} \quad (\text{B.7})$$

More details on the treatment of the zero mode can be found in [26] and many other places.